41 Attenuators

At the end of this chapter you should be able to:

- understand the function of an attenuator
- understand characteristic impedance and calculate for given values
- appreciate and calculate logarithmic ratios
- design symmetrical T and symmetrical π attenuators given required attenuation and characteristic impedance
- appreciate and calculate insertion loss
- determine iterative and image impedances for asymmetrical T and π networks
- appreciate and design the *L*-section attenuator
- calculate attenuation for two-port networks in cascade

41.1 Introduction

An **attenuator** is a device for introducing a specified loss between a signal source and a matched load without upsetting the impedance relationship necessary for matching. The loss introduced is constant irrespective of frequency; since reactive elements (*L* or *C*) vary with frequency, it follows that ideal attenuators are networks containing pure resistances. A fixed attenuator section is usually known as a 'pad'.

Attenuation is a reduction in the magnitude of a voltage or current due to its transmission over a line or through an attenuator. Any degree of attenuation may be achieved with an attenuator by suitable choice of resistance values but the input and output impedances of the pad must be such that the impedance conditions existing in the circuit into which it is connected are not disturbed. Thus an attenuator must provide the correct input and output impedances as well as providing the required attenuation.

Attenuation sections are made up of resistances connected as T or π arrangements (as introduced in Chapter 34).

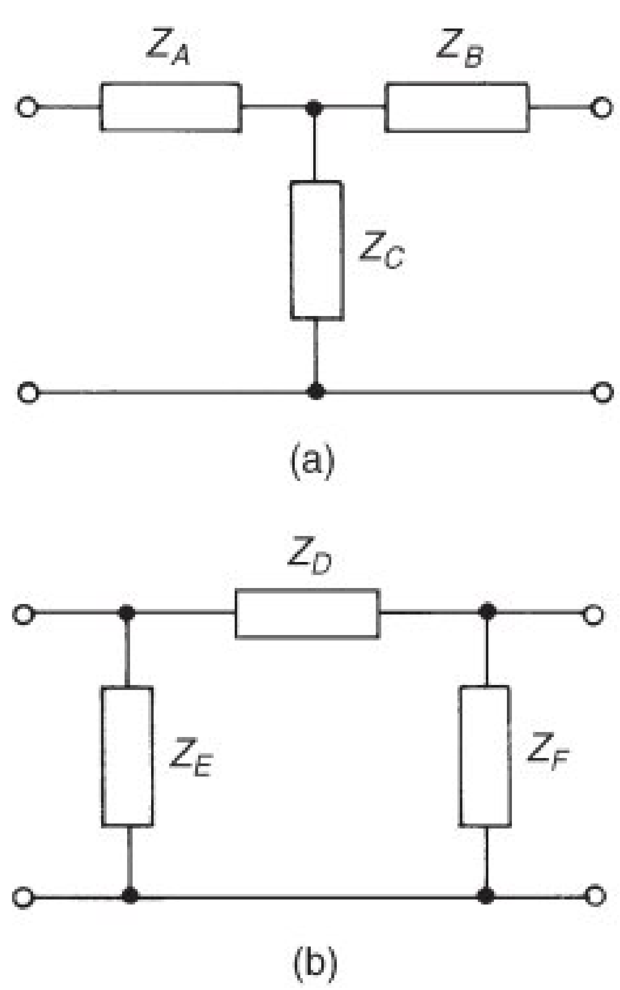


Figure 41.1 *(a) T-network, (b) π-network*

Two-port networks

Networks in which electrical energy is fed in at one pair of terminals and taken out at a second pair of terminals are called two-port networks. Thus an attenuator is a two-port network, as are transmission lines, transformers and electronic amplifiers. The network between the input port and the output port is a transmission network for which a known relationship exists between the input and output currents and voltages. If

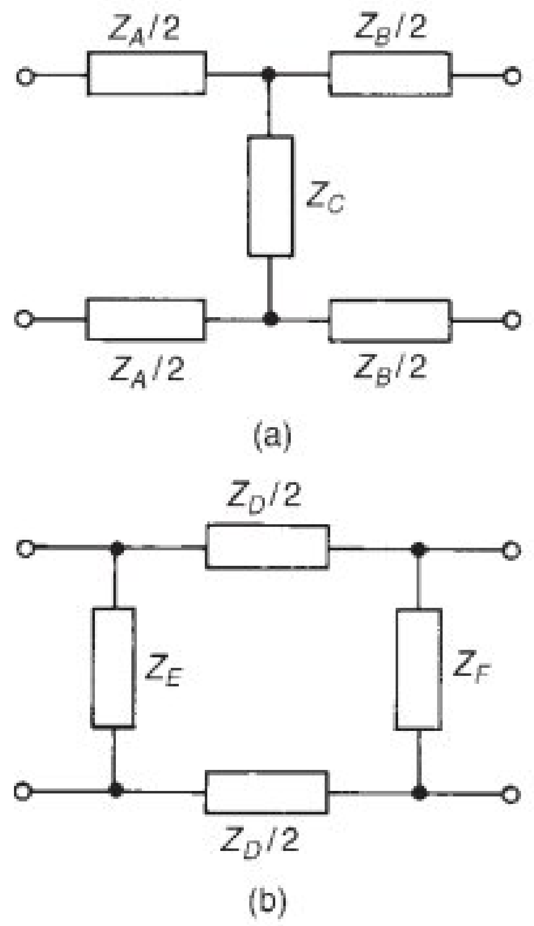


Figure 41.2 *(a) Balanced T-network, (b) Balanced π-network*

a network contains only passive circuit elements, such as in an attenuator, the network is said to be **passive**; if a network contains a source of e.m.f., such as in an electronic amplifier, the network is said to be **active**.

Figure 41.1(a) shows a T-network, which is termed **symmetrical** if $Z_A = Z_B$ and Figure 41.1(b) shows a π-network which is symmetrical if $Z_E = Z_F$. If $Z_A \neq Z_B$ in Figure 41.1(a) and $Z_E \neq Z_F$ in Figure 41.1(b), the sections are termed **asymmetrical**. Both networks shown have one common terminal, which may be earthed, and are therefore said to be **unbalanced.** The **balanced** form of the T-network is shown in Figure 41.2(a) and the balanced form of the π-network is shown in Figure 41.2(b).

Symmetrical T- and π-attenuators are discussed in Section 41.4 and asymmetrical attenuators are discussed in Sections 41.6 and 41.7. Before this it is important to understand the concept of characteristic impedance, which is explained generally in Section 41.2 (characteristic impedances will be used again in Chapter 44), and logarithmic units, discussed in Section 41.3. Another important aspect of attenuators, that of insertion loss, is discussed in Section 41.5. To obtain greater attenuation, sections may be connected in cascade, and this is discussed in Section 41.8.

41.2 Characteristic impedance

The input impedance of a network is the ratio of voltage to current (in complex form) at the input terminals. With a two-port network the input impedance often varies according to the load impedance across the output terminals. For any passive two-port network it is found that a particular value of load impedance can always be found which will produce an input impedance having the same value as the load impedance. This is called the **iterative impedance** for an asymmetrical network and its value depends on which pair of terminals is taken to be the input and which the output (there are thus two values of iterative impedance, one for each direction). For a symmetrical network there is only one value for the iterative impedance and this is called the **characteristic impedance** of the symmetrical two-port network. Let the characteristic impedance be denoted by Z_0. Figure 41.3 shows a **symmetrical *T*-network** terminated in an impedance Z_0.

Let the impedance 'looking-in' at the input port also be Z_0. Then $V_1/I_1 = Z_0 = V_2/I_2$ in Figure 41.3. From circuit theory,

$$Z_0 = \frac{V_1}{I_1} = Z_A + \frac{Z_B(Z_A + Z_0)}{Z_B + Z_A + Z_0}, \text{ since } (Z_A + Z_0) \text{ is in parallel with } Z_B,$$

$$= \frac{Z_A^2 + Z_AZ_B + Z_AZ_0 + Z_AZ_B + Z_BZ_0}{Z_A + Z_B + Z_0}$$

i.e. $$Z_0 = \frac{Z_A^2 + 2Z_AZ_B + Z_AZ_0 + Z_BZ_0}{Z_A + Z_B + Z_0}$$

Thus $$Z_0(Z_A + Z_B + Z_0) = Z_A^2 + 2Z_AZ_B + Z_AZ_0 + Z_BZ_0$$

$$Z_0Z_A + Z_0Z_B + Z_0^2 = Z_A^2 + 2Z_AZ_B + Z_AZ_0 + Z_BZ_0$$

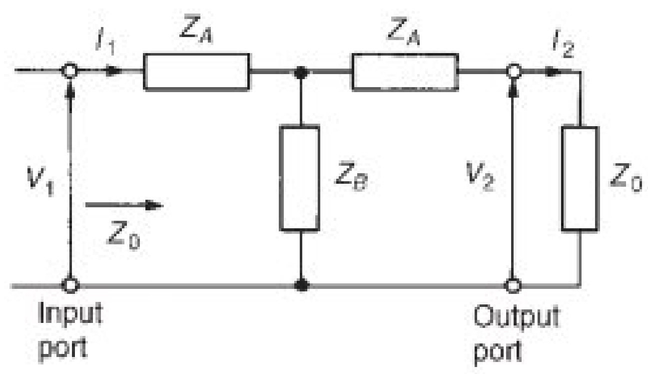


Figure 41.3

i.e., $Z_0^2 = Z_A^2 + 2Z_AZ_B$, from which

$$\text{characteristic impedance,} \quad \boxed{\mathbf{Z_0 = \sqrt{(Z_A^2 + 2Z_AZ_B)}}} \qquad (41.1)$$

If the output terminals of Figure 41.3 are open-circuited, then the open-circuit impedance, $Z_{OC} = Z_A + Z_B$. If the output terminals of Figure 41.3 are short-circuited, then the short-circuit impedance,

$$Z_{SC} = Z_A + \frac{Z_AZ_B}{Z_A + Z_B} = \frac{Z_A^2 + 2Z_AZ_B}{Z_A + Z_B}$$

Thus $\quad Z_{OC}Z_{SC} = (Z_A + Z_B)\left(\dfrac{Z_A^2 + 2Z_AZ_B}{Z_A + z_B}\right) = Z_A^2 + 2Z_AZ_B$

Comparing this with equation (41.1) gives

$$\boxed{\mathbf{Z_0 = \sqrt{(Z_{OC}Z_{SC})}},} \qquad (41.2)$$

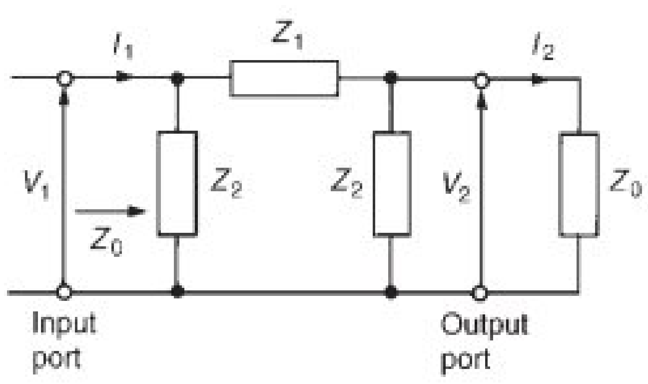


Figure 41.4

Figure 41.4 shows a symmetrical π-network terminated in an impedance Z_0.

If the impedance 'looking in' at the input port is also Z_0, then

$\dfrac{V_1}{I_1} = Z_0 = (Z_2)$ in parallel with [Z_1 in series with (Z_0 and Z_2) in parallel]

$= (Z_2)$ in parallel with $\left[Z_1 + \dfrac{Z_0Z_2}{Z_0 + Z_2}\right]$

$= (Z_2)$ in parallel with $\left[\dfrac{Z_1Z_0 + Z_1Z_2 + Z_0Z_2}{Z_0 + Z_2}\right]$

$$\text{i.e.,} \quad Z_0 = \frac{(Z_2)((Z_1Z_0 + Z_1Z_2 + Z_0Z_2)/(Z_0 + Z_2))}{Z_2 + ((Z_1Z_0 + Z_1Z_2 + Z_0Z_2)/(Z_0 + Z_2))}$$

$$= \frac{(Z_1Z_2Z_0 + Z_1Z_2^2 + Z_0Z_2^2)/(Z_0 + Z_2)}{(Z_2Z_0 + Z_2^2 + Z_1Z_0 + Z_1Z_2 + Z_0Z_2)/(Z_0 + Z_2)}$$

$$\text{i.e.} \quad Z_0 = \frac{Z_1Z_2Z_0 + Z_1Z_2^2 + Z_0Z_2^2}{Z_2^2 + 2Z_2Z_0 + Z_1Z_0 + Z_1Z_2}$$

Thus $\quad Z_0(Z_2^2 + 2Z_2Z_0 + Z_1Z_0 + Z_1Z_2) = Z_1Z_2Z_0 + Z_1Z_2^2 + Z_0Z_2^2$

$$2Z_2Z_0^2 + Z_1Z_0^2 = Z_1Z_2^2$$

from which

$$\boxed{\textbf{characteristic impedance, } \mathbf{Z_0 = \sqrt{\left(\frac{Z_1Z_2^2}{Z_1 + 2Z_2}\right)}}} \qquad (41.3)$$

If the output terminals of Figure 41.4 are open-circuited, then the open-circuit impedance,

$$Z_{OC} = \frac{Z_2(Z_1 + Z_2)}{Z_2 + Z_1 + Z_2} = \frac{Z_2(Z_1 + Z_2)}{Z_1 + 2Z_2}$$

If the output terminals of Figure 41.4 are short-circuited, then the short-circuit impedance,

$$Z_{SC} = \frac{Z_2 Z_1}{Z_1 + Z_2}$$

Thus

$$Z_{OC} Z_{SC} = \frac{Z_2(Z_1 + Z_2)}{(Z_1 + 2Z_2)} \left(\frac{Z_2 Z_1}{Z_1 + Z_2}\right) = \frac{Z_1 Z_2^2}{Z_1 + 2Z_2}$$

Comparing this expression with equation (41.3) gives

$$\mathbf{Z_0 = \sqrt{(Z_{OC} Z_{SC})},} \qquad (41.2')$$

which is the same as equation (41.2).

Thus the characteristic impedance Z_0 is given by $Z_0 = \sqrt{(Z_{OC} Z_{SC})}$ whether the network is a symmetrical T or a symmetrical π.

Equations (41.1) to (41.3) are used later in this chapter.

41.3 Logarithmic ratios

The ratio of two powers P_1 and P_2 may be expressed in logarithmic form as shown in Chapter 10.

Let P_1 be the input power to a system and P_2 the output power.

If logarithms to base 10 are used, then the ratio is said to be in **bels**, i.e., power ratio in bels $= \lg(P_2/P_1)$. The bel is a large unit and the **decibel (dB)** is more often used, where 10 decibels = 1 bel, i.e.,

$$\textbf{power ratio in decibels} = \mathbf{10 \lg \frac{P_2}{P_1}} \qquad (41.4)$$

For example:

P_2/P_1	Power ratio (dB)
1	$10 \lg 1 = 0$
100	$10 \lg 100 = +20$ (power gain)
$\frac{1}{10}$	$10 \lg \frac{1}{10} = -10$ (power loss or attenuation)

If **logarithms to base *e*** (i.e., natural or Napierian logarithms) are used, then the ratio of two powers is said to be in **nepers (Np)**, i.e.,

$$\boxed{\textbf{power ratio in nepers} = \frac{1}{2}\ln\frac{P_2}{P_1}} \qquad (41.5)$$

Thus when the power ratio $P_2/P_1 = 5$, the power ratio in nepers $= \frac{1}{2}\ln 5 = 0.805$ Np, and when the power ratio $P_2/P_1 = 0.1$, the power ratio in nepers $= \frac{1}{2}\ln 0.1 = -1.15$ Np.

The attenuation of filter sections and along a transmission line are of an exponential form and it is in such applications that the unit of the neper is used (see Chapters 42 and 44).

If the powers P_1 and P_2 refer to power developed in two equal resistors, R, then $P_1 = V_1^2/R$ and $P_2 = V_2^2/R$. Thus the ratio (from equation (41.4)) can be expressed, by the laws of logarithms, as

$$\textbf{ratio in decibels} = 10\lg\frac{P_2}{P_1} = 10\lg\left(\frac{V_2^2/R}{V_1^2/R}\right) = 10\lg\frac{V_2^2}{V_1^2}$$

$$= 10\lg\left(\frac{V_2}{V_1}\right)^2$$

i.e. $$\boxed{\textbf{ratio in decibels} = 20\lg\frac{V_2}{V_1}} \qquad (41.6)$$

Although this is really a power ratio, it is called the **logarithmic voltage ratio**.

Alternatively, (from equation (41.5)),

$$\text{ratio in nepers} = \frac{1}{2}\ln\frac{P_2}{P_1} = \frac{1}{2}\ln\left(\frac{V_2^2/R}{V_1^2/R}\right) = \frac{1}{2}\ln\left(\frac{V_2}{V_1}\right)^2$$

i.e., $$\boxed{\textbf{ratio in nepers} = \ln\frac{V_2}{V_1}} \qquad (41.7)$$

Similarly, if currents I_1 and I_2 in two equal resistors R give powers P_1 and P_2 then (from equation (41.4))

$$\text{ratio in decibels} = 10\lg\frac{P_2}{P_1} = 10\lg\left(\frac{I_2^2R}{I_1^2R}\right) = 10\lg\left(\frac{I_2}{I_1}\right)^2$$

i.e., $$\boxed{\textbf{ratio in decibels} = 20\lg\frac{I_2}{I_1}} \qquad (41.8)$$

Alternatively (from equation (41.5)),

$$\text{ratio in nepers} = \frac{1}{2}\ln\frac{P_2}{P_1} = \frac{1}{2}\ln\left(\frac{I_2^2R}{I_1^2R}\right)^2 = \frac{1}{2}\ln\left(\frac{I_2}{I_1}\right)^2$$

i.e., **ratio in nepers** $= \ln \dfrac{I_2}{I_1}$ (41.9)

In equations (41.4) to (41.9) the output-to-input ratio has been used. However, the input-to-output ratio may also be used. For example, in equation (41.6), the output-to-input voltage ratio is expressed as $20 \lg(V_2/V_1)$ dB. Alternatively, the input-to-output voltage ratio may be expressed as $20 \lg(V_1/V_2)$ dB, the only difference in the values obtained being a difference in sign.

If $20 \lg(V_2/V_1) = 10$ dB, say, then $20 \lg(V_1/V_2) = -10$ dB. Thus if an attenuator has a voltage input V_1 of 50 mV and a voltage output V_2 of 5 mV, the voltage ratio V_2/V_1 is 5/50 or 1/10. Alternatively, this may be expressed as **'an attenuation of 10'**, i.e., $V_1/V_2 = 10$.

Problem 1. The ratio of output power to input power in a system is

(a) 2 (b) 25 (c) 1000 and (d) $\frac{1}{100}$

Determine the power ratio in each case (i) in decibels and (ii) in nepers.

(i) From equation (41.4), power ratio in decibels $= 10 \lg(P_2/P_1)$.

(a) When $P_2/P_1 = 2$, power ratio $= 10 \lg 2 = \mathbf{3\ dB}$
(b) When $P_2/P_1 = 25$, power ratio $= 10 \lg 25 = \mathbf{14\ dB}$
(c) When $P_2/P_1 = 1000$, power ratio $= 10 \lg 1000 = \mathbf{30\ dB}$
(d) When $P_2/P_1 = \frac{1}{100}$, power ratio $= 10 \lg \frac{1}{100} = \mathbf{-20\ dB}$

(ii) From equation (41.5), power ratio in nepers $= \frac{1}{2} \ln(P_2/P_1)$.

(a) When $P_2/P_1 = 2$, power ratio $= \frac{1}{2} \ln 2 = \mathbf{0.347\ Np}$
(b) When $P_2/P_1 = 25$, power ratio $= \frac{1}{2} \ln 25 = \mathbf{1.609\ Np}$
(c) When $P_2/P_1 = 1000$, power ratio $= \frac{1}{2} \ln 1000 = \mathbf{3.454\ Np}$
(d) When $P_2/P_1 = \frac{1}{100}$, power ratio $= \frac{1}{2} \ln \frac{1}{100} = \mathbf{-2.303\ Np}$

The power ratios in (a), (b) and (c) represent power gains, since the ratios are positive values; the power ratio in (d) represents a power loss or attenuation, since the ratio is a negative value.

Problem 2. 5% of the power supplied to a cable appears at the output terminals. Determine the attenuation in decibels.

If P_1 = input power and P_2 = output power, then

$$\frac{P_2}{P_1} = \frac{5}{100} = 0.05$$

From equation (41.4), power ratio in decibels

$$= 10\lg(P_2/P_1) = 10\lg 0.05 = -13 \text{ dB}.$$

Hence the attenuation (i.e., power loss) is 13 dB.

> Problem 3. An amplifier has a gain of 15 dB. If the input power is 12 mW, determine the output power.

From equation (41.4), decibel power ratio $= 10\lg(P_2/P_1)$. Hence $15 = 10\lg(P_2/12)$, where P_2 is the output power in milliwatts.

$$1.5 = \lg\left(\frac{P_2}{12}\right)$$

$$\frac{P_2}{12} = 10^{1.5}$$

from the definition of a logarithm. Thus the output power,

$$P_2 = 12(10)^{1.5} = \mathbf{379.5\ mW}$$

> Problem 4. The current output of an attenuator is 50 mA. If the current ratio of the attenuator is -1.32 Np, determine (a) the current input and (b) the current ratio expressed in decibels. Assume that the input and load resistances of the attenuator are equal.

(a) From equation (41.9), current ratio in nepers $= \ln(I_2/I_1)$. Hence $-1.32 = \ln(50/I_1)$, where I_1 is the input current in mA.

$$e^{-1.32} = \frac{50}{I_1}$$

from which, **current input**, $\boldsymbol{I_1} = \dfrac{50}{e^{-1.32}} = 50e^{1.32} = \mathbf{187.2\ mA}$

(b) From equation (41.8),

$$\text{current ratio in decibels} = 20\lg\frac{I_2}{I_1} = 20\lg\left(\frac{50}{187.2}\right)$$

$$= \mathbf{-11.47\ dB}$$

Further problems on logarithmic ratios may be found in Section 41.9, problems 1 to 5, page 785.

41.4 Symmetrical T-and π-attenuators

(a) Symmetrical T-attenuator

As mentioned in Section 41.1, the ideal attenuator is made up of pure resistances. A symmetrical T-pad attenuator is shown in Figure 41.5 with a termination R_0 connected as shown. From equation (41.1),

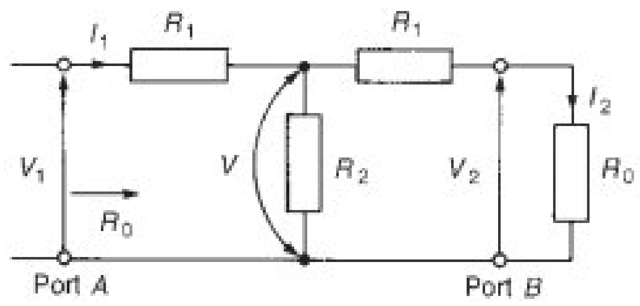


Figure 41.5 *Symmetrical T-pad attenuator*

$$\boldsymbol{R_0 = \sqrt{(R_1^2 + 2R_1R_2)}} \qquad (41.10)$$

and from equation (41.2) $$\boldsymbol{R_0 = \sqrt{(R_{OC}R_{SC})}} \qquad (41.11)$$

With resistance R_0 as the termination, the input resistance of the pad will also be equal to R_0. If the terminating resistance R_0 is transferred to port A then the input resistance looking into port B will again be R_0.

The pad is therefore symmetrical in impedance in both directions of connection and may thus be inserted into a network whose impedance is also R_0. The value of R_0 is the characteristic impedance of the section.

As stated in Section 41.3, attenuation may be expressed as a voltage ratio V_1/V_2 (see Figure 41.5) or quoted in decibels as $20\lg(V_1/V_2)$ or, alternatively, as a power ratio as $10\lg(P_1/P_2)$. If a T-section is symmetrical, i.e., the terminals of the section are matched to equal impedances, then

$$10\lg\frac{P_1}{P_2} = 20\lg\frac{V_1}{V_2} = 20\lg\frac{I_1}{I_2}$$

since $R_{IN} = R_{LOAD} = R_0$, i.e.,

$$10\lg\frac{P_1}{P_2} = 10\lg\left(\frac{V_1}{V_2}\right)^2 = 10\lg\left(\frac{I_1}{I_2}\right)^2$$

from which $$\frac{P_1}{P_2} = \left(\frac{V_1}{V_2}\right)^2 = \left(\frac{I_1}{I_2}\right)^2$$

or $$\sqrt{\left(\frac{P_1}{P_2}\right)} = \left(\frac{V_1}{V_2}\right) = \left(\frac{I_1}{I_2}\right)$$

Let $N = V_1/V_2$ or I_1/I_2 or $\sqrt{(P_1/P_2)}$, where N is the attenuation. In Section 41.5, page 772, it is shown that, for a matched network, i.e., one terminated in its characteristic impedance, N is in fact the insertion loss ratio. (Note that in an asymmetrical network, only the expression $N = \sqrt{(P_1/P_2)}$ may be used — see Section 41.7 on the L-section attenuator)

From Figure 41.5,

$$\text{current} \quad I_1 = \frac{V_1}{R_0}$$

$$\text{Voltage} \quad V = V_1 - I_1R_1 = V_1 - \left(\frac{V_1}{R_0}\right)R_1$$

$$\text{i.e.,} \qquad V = V_1\left(1 - \frac{R_1}{R_0}\right)$$

$$\text{Voltage } V_2 = \left(\frac{R_0}{R_1 + R_0}\right)V \text{ by voltage division}$$

$$\text{i.e.,} \quad V_2 = \left(\frac{R_0}{R_1 + R_0}\right) V_1 \left(1 - \frac{R_1}{R_0}\right)$$

$$= V_1 \left(\frac{R_0}{R_1 + R_0}\right)\left(\frac{R_0 - R_1}{R_0}\right)$$

Hence $$\frac{V_2}{V_1} = \frac{R_0 - R_1}{R_0 + R_1} \text{ or } \frac{V_1}{V_2} = N = \frac{R_0 + R_1}{R_0 - R_1} \qquad (41.12)$$

From equation (41.12) and also equation (41.10), it is possible to derive expressions for R_1 and R_2 in terms of N and R_0, thus enabling an attenuator to be designed to give a specified attenuation and to be matched symmetrically into the network. From equation (41.12),

$$N(R_0 - R_1) = R_0 + R_1$$

$$NR_0 - NR_1 = R_0 + R_1$$

$$NR_0 - R_0 = R_1 + NR_1$$

$$R_0(N - 1) = R_1(1 + N)$$

from which $$\boxed{\boldsymbol{R_1 = R_0 \frac{(N - 1)}{(N + 1)}}} \qquad (41.13)$$

From equation (41.10), $R_0 = \sqrt{(R_1^2 + 2R_1R_2)}$ i.e., $R_0^2 = R_1^2 + 2R_1R_2$,

from which, $$R_2 = \frac{R_0^2 - R_1^2}{2R_1}$$

Substituting for R_1 from equation (41.13) gives

$$R_2 = \frac{R_0^2 - [R_0(N - 1)/(N + 1)]^2}{2[R_0(N - 1)/(N + 1)]}$$

$$= \frac{[R_0^2(N + 1)^2 - R_0^2(N - 1)^2]/(N + 1)^2}{2R_0(N - 1)/(N + 1)}$$

$$\text{i.e.,} \quad R_2 = \frac{R_0^2[(N + 1)^2 - (N - 1)^2]}{2R_0(N - 1)(N + 1)}$$

$$= \frac{R_0[(N^2 + 2N + 1) - (N^2 - 2N + 1)]}{2(N^2 - 1)}$$

$$= \frac{R_0(4N)}{2(N^2 - 1)}$$

Hence $$\boxed{\boldsymbol{R_2 = R_0 \left(\frac{2N}{N^2 - 1}\right)}} \qquad (41.14)$$

Thus if the characteristic impedance R_0 and the attenuation N $(= V_1/V_2)$ are known for a symmetrical T-network then values of R_1 and R_2 may be

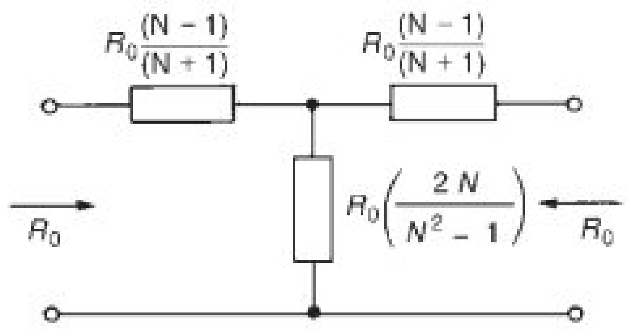


Figure 41.6

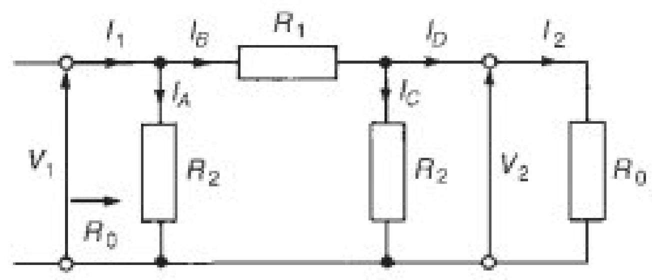


Figure 41.7 *Symmetrical π-attenuator*

calculated. Figure 41.6 shows a T-pad attenuator having input and output impedances of R_0 with resistances R_1 and R_2 expressed in terms of R_0 and N.

(b) Symmetrical π-attenuator

A symmetrical π-attenuator is shown in Figure 41.7 terminated in R_0. From equation (41.3),

$$\text{characteristic impedance} \quad \boxed{\boldsymbol{R_0 = \sqrt{\left(\frac{R_1 R_2^2}{R_1 + 2R_2}\right)}}} \tag{41.15}$$

$$\text{and from equation (41.2}'\text{),} \quad \boxed{\boldsymbol{R_0 = \sqrt{(R_{OC} R_{SC})}}} \tag{41.16}$$

Given the attenuation factor $N = \dfrac{V_1}{V_2}\left(= \dfrac{I_1}{I_2}\right)$

and the characteristic impedance R_0, it is possible to derive expressions for R_1 and R_2, in a similar way to the T-pad attenuator, to enable a π-attenuator to be effectively designed.

Since $N = V_1/V_2$ then $V_2 = V_1/N$. From Figure 41.7,

current $I_1 = I_A + I_B$ and current $I_B = I_C + I_D$. Thus

$$\text{current } I_1 = \frac{V_1}{R_0} = I_A + I_C + I_D$$

$$= \frac{V_1}{R_2} + \frac{V_2}{R_2} + \frac{V_2}{R_0} = \frac{V_1}{R_2} + \frac{V_1}{NR_2} + \frac{V_1}{NR_0}$$

since $V_2 = V_1/N$, i.e.,

$$\frac{V_1}{R_0} = V_1\left(\frac{1}{R_2} + \frac{1}{NR_2} + \frac{1}{NR_0}\right)$$

$$\text{Hence} \quad \frac{1}{R_0} = \frac{1}{R_2} + \frac{1}{NR_2} + \frac{1}{NR_0}$$

$$\frac{1}{R_0} - \frac{1}{NR_0} = \frac{1}{R_2} + \frac{1}{NR_2}$$

$$\frac{1}{R_0}\left(1 - \frac{1}{N}\right) = \frac{1}{R_2}\left(1 + \frac{1}{N}\right)$$

$$\frac{1}{R_0}\left(\frac{N-1}{N}\right) = \frac{1}{R_2}\left(\frac{N+1}{N}\right)$$

$$\text{Thus} \quad \boxed{\boldsymbol{R_2 = R_0 \frac{(N+1)}{(N-1)}}} \tag{41.17}$$

From Figure 41.7, current $I_1 = I_A + I_B$, and since the p.d. across R_1 is $(V_1 - V_2)$,

$$\frac{V_1}{R_0} = \frac{V_1}{R_2} + \frac{V_1 - V_2}{R_1}$$

$$\frac{V_1}{R_0} = \frac{V_1}{R_2} + \frac{V_1}{R_1} - \frac{V_2}{R_1}$$

$$\frac{V_1}{R_0} = \frac{V_1}{R_2} + \frac{V_1}{R_1} - \frac{V_1}{NR_1} \quad \text{since } V_2 = V_1/N$$

$$\frac{1}{R_0} = \frac{1}{R_2} + \frac{1}{R_1} - \frac{1}{NR_1}$$

$$\frac{1}{R_0} - \frac{1}{R_2} = \frac{1}{R_1}\left(1 - \frac{1}{N}\right)$$

$$\frac{1}{R_0} - \frac{(N-1)}{R_0(N+1)} = \frac{1}{R_1}\left(\frac{N-1}{N}\right) \quad \text{from equation (41.17),}$$

$$\frac{1}{R_0}\left(1 - \frac{N-1}{N+1}\right) = \frac{1}{R_1}\left(\frac{N-1}{N}\right)$$

$$\frac{1}{R_0}\left(\frac{(N+1)-(N-1)}{(N+1)}\right) = \frac{1}{R_1}\left(\frac{N-1}{N}\right)$$

$$\frac{1}{R_0}\left(\frac{2}{N+1}\right) = \frac{1}{R_1}\left(\frac{N-1}{N}\right)$$

$$R_1 = R_0\left(\frac{N-1}{N}\right)\left(\frac{N+1}{2}\right)$$

Hence

$$\boxed{\mathbf{R_1 = R_0\left(\frac{N^2-1}{2N}\right)}} \qquad (41.18)$$

Figure 41.8 shows a π-attenuator having input and output impedances of R_0 with resistances R_1 and R_2 expressed in terms of R_0 and N.

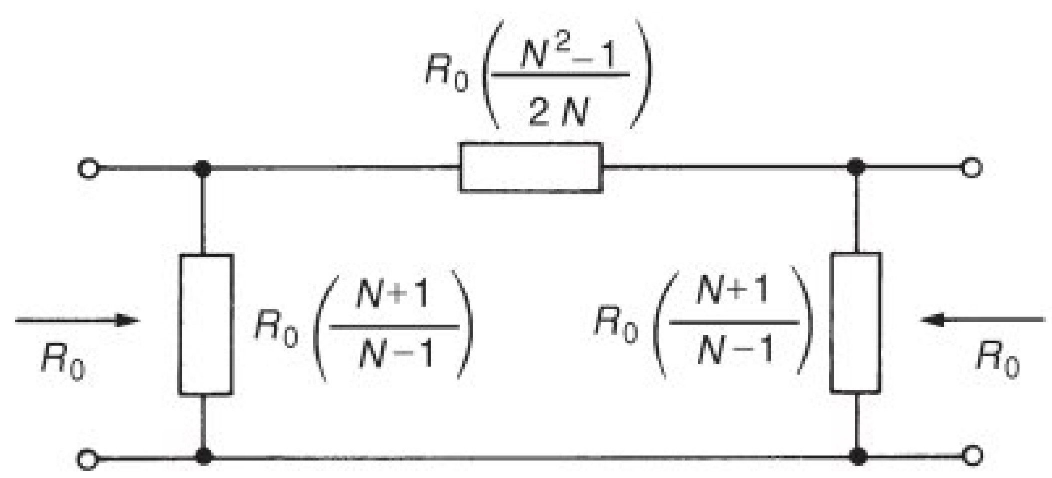


Figure 41.8

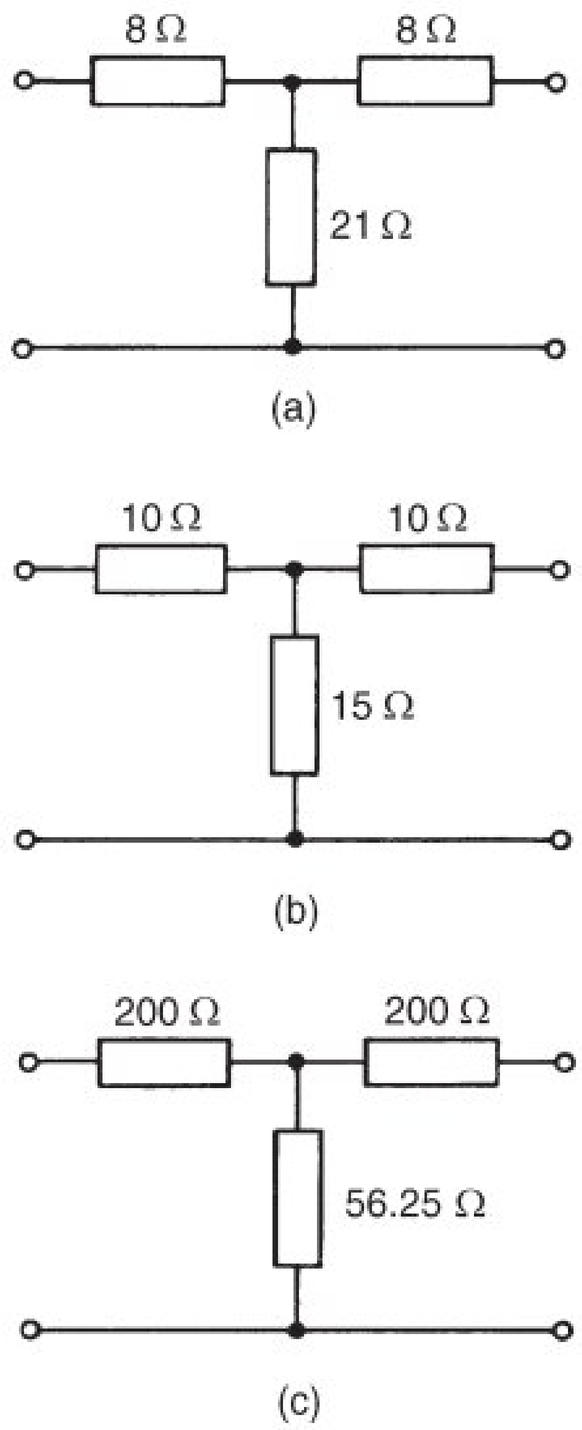


Figure 41.9

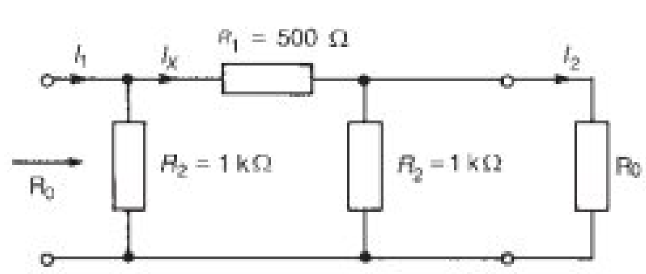


Figure 41.10

There is no difference in the functions of the T- and π-attenuator pads and either may be used in a particular situation.

> Problem 5. Determine the characteristic impedance of each of the attenuator sections shown in Figure 41.9.

From equation (41.10), for a T-section attenuator the characteristic impedance,

$$R_0 = \sqrt{(R_1^2 + 2R_1R_2)}.$$

(a) $R_0 = \sqrt{(8^2 + (2)(8)(21))} = \sqrt{400} = \mathbf{20\ \Omega}$

(b) $R_0 = \sqrt{(10^2 + (2)(10)(15))} = \sqrt{400} = \mathbf{20\ \Omega}$

(c) $R_0 = \sqrt{(200^2 + (2)(200)(56.25))} = \sqrt{62500} = \mathbf{250\ \Omega}$

It is seen that the characteristic impedance of parts (a) and (b) is the same. In fact, there are numerous combinations of resistances R_1 and R_2 which would give the same value for the characteristic impedance.

> Problem 6. A symmetrical π-attenuator pad has a series arm of 500 Ω resistance and each shunt arm of 1 kΩ resistance. Determine (a) the characteristic impedance, and (b) the attenuation (in dB) produced by the pad.

The π-attenuator section is shown in Figure 41.10 terminated in its characteristic impedance, R_0.

(a) From equation (41.15), for a symmetrical π-attenuator section,

$$\text{characteristic impedance, } R_0 = \sqrt{\left(\frac{R_1R_2^2}{R_1 + 2R_2}\right)}$$

$$\text{Hence} \quad R_0 = \sqrt{\left[\frac{(500)(1000)^2}{500 + 2(1000)}\right]} = 447\ \Omega$$

(b) Attenuation $= 20\lg(I_1/I_2)$ dB. From Figure 41.10,

$$\text{current } I_X = \left(\frac{R_2}{R_2 + R_1 + (R_2R_0/(R_2 + R_0))}\right)(I_1),$$

by current division

$$\text{i.e.,} \quad I_X = \left(\frac{1000}{1000 + 500 + ((1000)(447)/(1000 + 447))}\right) I_1$$

$$= 0.553 I_1$$

$$\text{and current } I_2 = \left(\frac{R_2}{R_2 + R_0}\right) I_X = \left(\frac{1000}{1000 + 447}\right) I_X = 0.691 I_X$$

Hence $I_2 = 0.691(0.553I_1) = 0.382I_1$ and $I_1/I_2 = 1/0.382$

$= 2.617$. Thus

attenuation $= 20\lg 2.617 =$ **8.36 dB**

(Alternatively, since $I_1/I_2 = N$, then the formula

$$R_2 = R_0\left(\frac{N+1}{N-1}\right)$$

may be transposed for N, from which **attenuation** $=$ **20 lg** $\boldsymbol{N}$.)

> Problem 7. For each of the attenuator networks shown in Figure 41.11, determine (a) the input resistance when the output port is open-circuited, (b) the input resistance when the output port is short-circuited, and (c) the characteristic impedance.

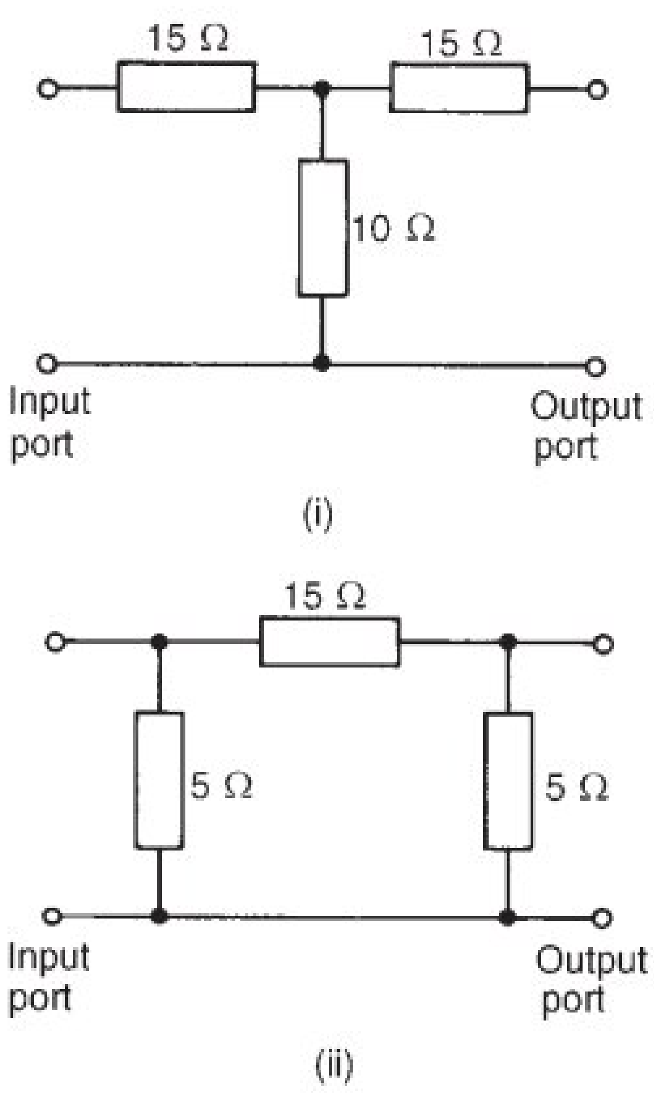


Figure 41.11

(i) For the T-network shown in Figure 41.11(i):

(a) $R_{OC} = 15 + 10 =$ **25 Ω**

(b) $R_{SC} = 15 + \dfrac{10 \times 15}{10 + 15} = 15 + 6 =$ **21 Ω**

(c) From equation (41.11), $R_0 = \sqrt{(R_{OC}R_{SC})} = \sqrt{[(25)(21)]}$
$=$ **22.9 Ω**

(Alternatively, from equation (41.10),

$R_0 = \sqrt{(R_1^2 + 2R_1R_2)} = \sqrt{(15^2 + (2)(15)(10))} =$ **22.9 Ω**)

(ii) For the π-network shown in Figure 41.11(ii):

(a) $R_{OC} = \dfrac{5 \times (15+5)}{5 + (15+5)} = \dfrac{100}{25} =$ **4 Ω**

(b) $R_{SC} = \dfrac{5 \times 15}{5 + 15} = \dfrac{75}{20} =$ **3.75 Ω**

(c) From equation (41.16),

$R_0 = \sqrt{(R_{OC}R_{SC})}$ as for a T-network

$= \sqrt{[(4)(3.75)]} = \sqrt{15} =$ **3.87 Ω**

(Alternatively, from equation (41.15),

$$R_0 = \sqrt{\left(\frac{R_1R_2^2}{R_1 + 2R_2}\right)} = \sqrt{\left(\frac{15(5)^2}{15 + 2(5)}\right)} = \mathbf{3.87\ \Omega})$$

> Problem 8. Design a T-section symmetrical attenuator pad to provide a voltage attenuation of 20 dB and having a characteristic impedance of 600 Ω.

Voltage attenuation in decibels $= 20\lg(V_1/V_2)$.

Attenuation, $N = V_1/V_2$, hence $20 = 20\lg N$, from which $N = 10$.

Characteristic impedance, $R_0 = 600\ \Omega$

From equation (41.13),

$$\text{resistance } R_1 = \frac{R_0(N-1)}{(N+1)} = \frac{600(10-1)}{(10+1)} = \mathbf{491\ \Omega}$$

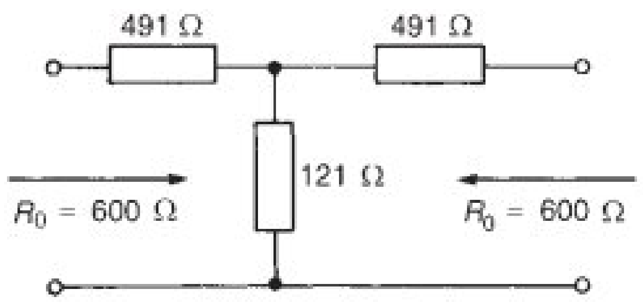


Figure 41.12

From equation (41.14),

$$\text{resistance } R_2 = R_0\left(\frac{2N}{N^2-1}\right) = 600\left(\frac{(2)(10)}{10^2-1}\right) = \mathbf{121\ \Omega}$$

Thus the T-section attenuator shown in Figure 41.12 has a voltage attenuation of 20 dB and a characteristic impedance of 600 Ω.
(Check: From equation (41.10)),

$$R_0 = \sqrt{(R_1^2 + 2R_1R_2)} = \sqrt{[491^2 + 2(491)(121)]} = 600\ \Omega)$$

Problem 9. Design a π-section symmetrical attenuator pad to provide a voltage attenuation of 20 dB and having a characteristic impedance of 600 Ω.

From problem 8, $N = 10$ and $R_0 = 600\ \Omega$

From equation (41.18),

$$\text{resistance } R_1 = R_0\left(\frac{N^2-1}{2N}\right) = 600\left(\frac{10^2-1}{(2)(10)}\right)$$

$$= \mathbf{2970\ \Omega} \text{ or } \mathbf{2.97\ k\Omega}$$

From equation (41.17),

$$R_2 = R_0\left(\frac{N+1}{N-1}\right) = 600\left(\frac{10+1}{10-1}\right) = \mathbf{733\ \Omega}$$

Thus the π-section attenuator shown in Figure 41.13 has a voltage attenuation of 20 dB and a characteristic impedance of 600 Ω.

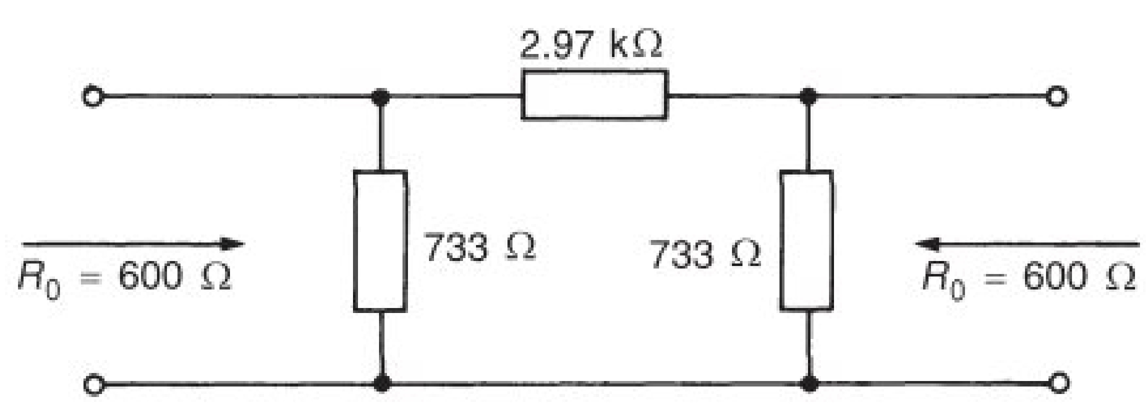


Figure 41.13

(Check: From equation (41.15),

$$R_0 = \sqrt{\left(\frac{R_1 R_2^2}{R_1 + 2R_2}\right)} = \sqrt{\left(\frac{(2970)(733)^2}{2970 + (2)(733)}\right)} = 600\ \Omega)$$

Further problems on symmetrical T- and π-attenuators may be found in Section 41.9, problems 6 to 15, page 785.

41.5 Insertion loss

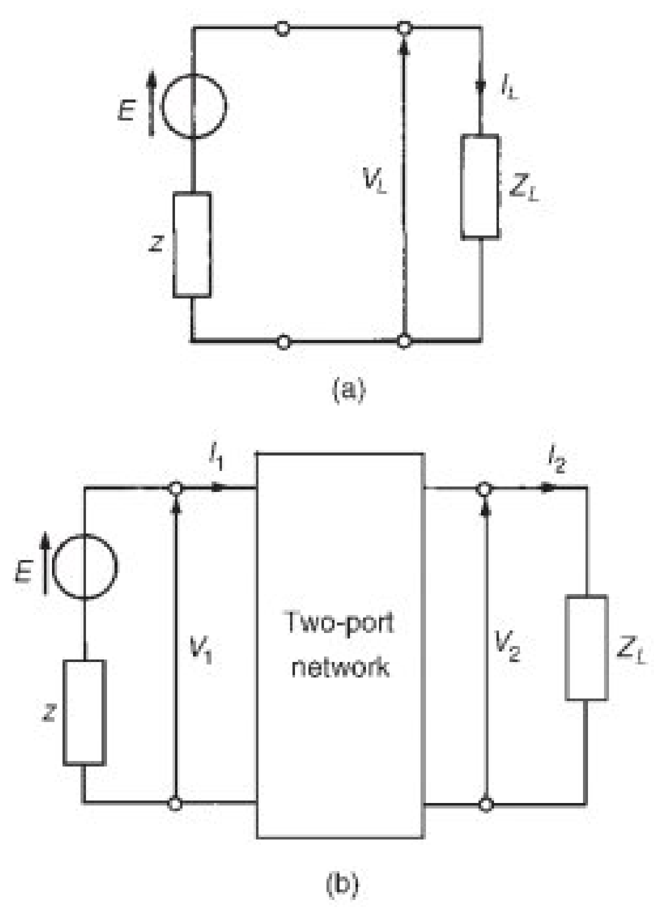


Figure 41.14

Figure 41.14(a) shows a generator E connected directly to a load Z_L. Let the current flowing be I_L and the p.d. across the load V_L. z is the internal impedance of the source.

Figure 41.14(b) shows a two-port network connected between the generator E and load Z_L.

The current through the load, shown as I_2, and the p.d. across the load, shown as V_2, will generally be less than current I_L and voltage V_L of Figure 41.14(a), as a result of the insertion of the two-port network between generator and load.

The **insertion loss ratio, A_L**, is defined as

$$A_L = \frac{\text{voltage across load when connected directly to the generator}}{\text{voltage across load when the two-port network is connected}}$$

i.e., $$\boxed{A_L = V_L/V_2 = I_L/I_2} \qquad (41.19)$$

since $V_L = I_L Z_L$ and $V_2 = I_2 Z_L$. Since both V_L and V_2 refer to p.d.'s across the same impedance Z_L, the insertion loss ratio may also be expressed (from Section 41.3) as

$$\boxed{\textbf{insertion loss ratio} = 20\lg\left(\frac{V_L}{V_2}\right)\ \textbf{dB or } 20\lg\left(\frac{I_L}{I_2}\right)\ \textbf{dB}} \qquad (41.20)$$

When the two-port network is terminated in its characteristic impedance Z_0 the network is said to be **matched**. In such circumstances the input impedance is also Z_0, thus the insertion loss is simply the ratio of input to output voltage (i.e., V_1/V_2). Thus, **for a network terminated in its characteristic impedance**,

$$\boxed{\textbf{insertion loss} = 20\ \lg\left(\frac{V_1}{V_2}\right)\textbf{dB or } 20\ \lg\left(\frac{I_1}{I_2}\right)\textbf{dB}} \qquad (41.21)$$

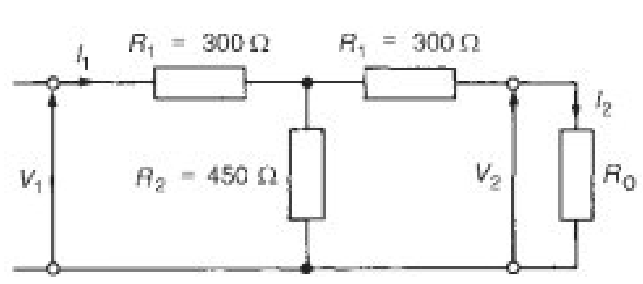


Figure 41.15

Problem 10. The attenuator shown in Figure 41.15 feeds a matched load. Determine (a) the characteristic impedance R_0, and (b) the insertion loss in decibels.

(a) From equation (41.10), the characteristic impedance of a symmetric T-pad attenuator is given by

$$R_0 = \sqrt{(R_1^2 + 2R_1R_2)} = \sqrt{[300^2 + 2(300)(450)]} = \mathbf{600\ \Omega}.$$

(b) Since the T-network is terminated in its characteristic impedance, then from equation (41.21),

$$\text{insertion loss} = 20\lg(V_1/V_2)\ \text{dB or}\ 20\lg(I_1/I_2)\ \text{dB}.$$

By current division in Figure 41.15,

$$I_2 = \left(\frac{R_2}{R_2 + R_1 + R_0}\right)(I_1)$$

Hence

$$\textbf{insertion loss} = 20\lg\frac{I_1}{I_2} = 20\lg\left(\frac{I_1}{(R_2/(R_2+R_1+R_0))I_1}\right)$$
$$= 20\lg\left(\frac{R_2+R_1+R_0}{R_2}\right)$$
$$= 20\lg\left(\frac{450+300+600}{450}\right)$$
$$= 20\lg 3 = \mathbf{9.54\ dB}$$

Problem 11. A 0–3 kΩ rheostat is connected across the output of a signal generator of internal resistance 500 Ω. If a load of 2 kΩ is connected across the rheostat, determine the insertion loss at a tapping of (a) 2 kΩ, (b) 1 kΩ.

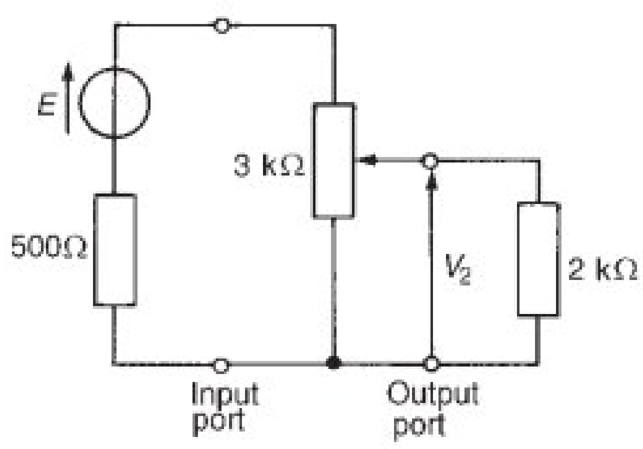


Figure 41.16

The circuit diagram is shown in Figure 41.16. Without the rheostat in the circuit the voltage across the 2 kΩ load, V_L (see Figure 41.17), is given by

$$V_L = \left(\frac{2000}{2000+500}\right)E = 0.8\ \text{E}$$

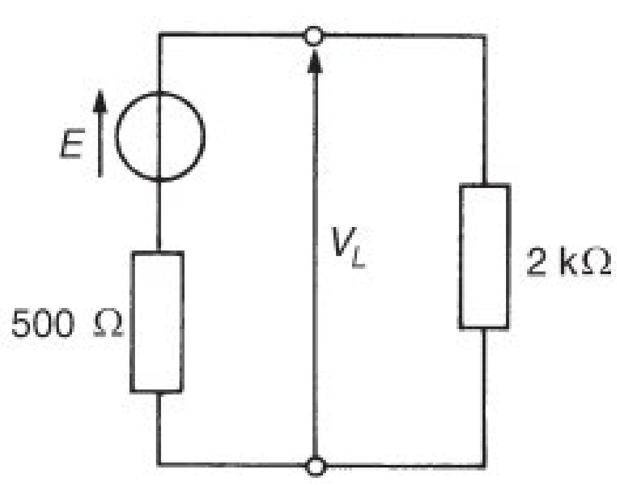


Figure 41.17

(a) With the 2 kΩ tapping, the network of Figure 41.16 may be redrawn as shown in Figure 41.18, which in turn is simplified as shown in Figure 41.19. From Figure 41.19,

$$\text{voltage } V_2 = \left(\frac{1000}{1000+1000+500}\right)E = 0.4\ \text{E}$$

Hence, from equation (41.19), insertion loss ratio,

$$A_L = \frac{V_L}{V_2} = \frac{0.8E}{0.4E} = \mathbf{2}$$

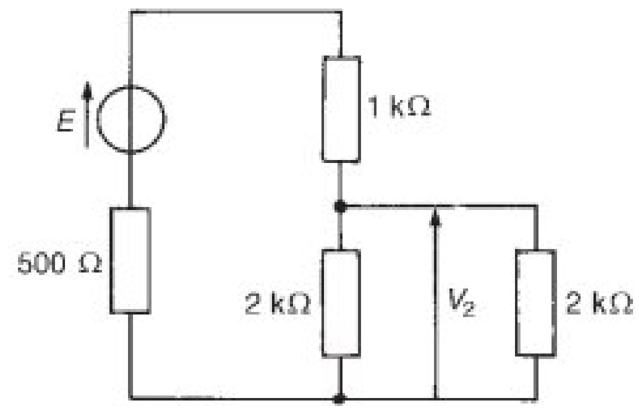


Figure 41.18

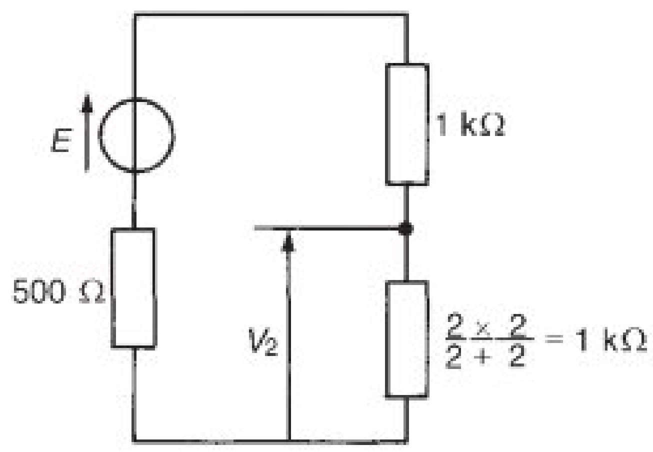


Figure 41.19

or, from equation (41.20),

$$\text{insertion loss} = 20 \lg(V_L/V_2) = 20 \lg 2 = \mathbf{6.02\ dB}$$

(b) With the 1 kΩ tapping, voltage V_2 is given by

$$V_2 = \left(\frac{(1000 \times 2000)/(1000 + 2000)}{((1000 \times 2000)/(1000 + 2000)) + 2000 + 500} \right) E$$

$$= \left(\frac{666.7}{666.7 + 2000 + 500} \right) E = 0.211\ \text{E}$$

Hence, from equation (41.19),

$$\text{insertion loss ratio } A_L = \frac{V_L}{V_2} = \frac{0.8E}{0.211E} = \mathbf{3.79}$$

or, from equation (41.20),

$$\text{insertion loss in decibels} = 20 \lg \left(\frac{V_L}{V_2} \right) = 20 \lg 3.79$$

$$= \mathbf{11.57\ dB}$$

(Note that the insertion loss is not doubled by halving the tapping.)

Problem 12. A symmetrical π-attenuator pad has a series arm of resistance 1000 Ω and shunt arms each of 500 Ω. Determine (a) its characteristic impedance, and (b) the insertion loss (in decibels) when feeding a matched load.

The π-attenuator pad is shown in Figure 41.20, terminated in its characteristic impedance, R_0.

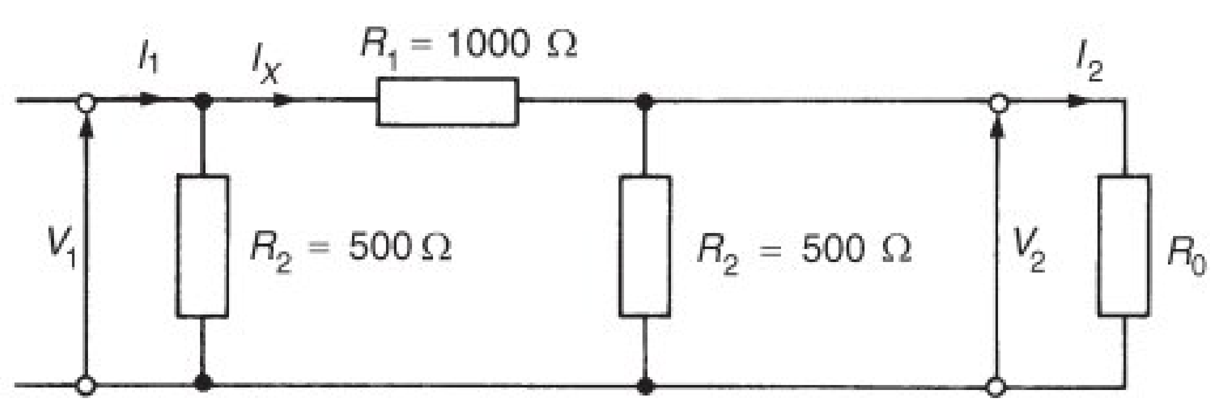


Figure 41.20

(a) From equation (41.15), the characteristic impedance of a symmetrical attenuator is given by

$$R_0 = \sqrt{\left(\frac{R_1 R_2^2}{R_1 + 2R_2} \right)} = \sqrt{\left(\frac{(1000)(500)^2}{1000 + 2(500)} \right)} = \mathbf{354\ \Omega}$$

(b) Since the attenuator network is feeding a matched load, from equation (41.21),

$$\text{insertion loss} = 20\lg\left(\frac{V_1}{V_2}\right)\ \text{dB} = 20\ \lg\left(\frac{I_1}{I_2}\right)\ \text{dB}$$

From Figure 41.20, by current division,

$$\text{current } I_X = \left\{\frac{R_2}{R_2 + R_1 + (R_2R_0/(R_2+R_0))}\right\}(I_1)$$

$$\text{and current } I_2 = \left(\frac{R_2}{R_2+R_0}\right) I_X$$

$$= \left(\frac{R_2}{R_2+R_0}\right)\left(\frac{R_2}{R_2 + R_1 + (R_2R_0/(R_2+R_0))}\right) I_1$$

i.e.,

$$I_2 = \left(\frac{500}{500+354}\right)\left(\frac{500}{500+1000+((500)(354)/(500+354))}\right) I_1$$

$$= (0.5855)(0.2929)I_1 = 0.1715I_1$$

Hence $I_1/I_2 = 1/0.1715 = 5.83$

Thus the insertion loss in decibels $= 20\lg(I_1/I_2)$

$= 20\lg 5.83 =$ **15.3 dB**

Further problems on insertion loss may be found in Section 41.9, problems 16 to 18, page 786.

41.6 Asymmetrical T- and π-sections

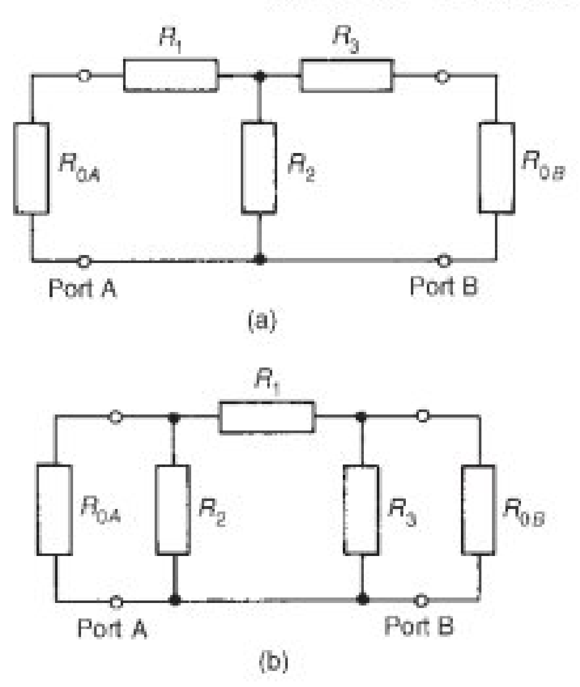


Figure 41.21 *(a) Asymmetrical T-pad section, (b) Asymmetrical π-section*

Figure 41.21(a) shows an asymmetrical T-pad section where resistance $R_1 \neq R_3$. Figure 41.21(b) shows an asymmetrical π-section where $R_2 \neq R_3$.

When viewed from port A, in each of the sections, the output impedance is R_{OB}; when viewed from port B, the input impedance is R_{OA}. Since the sections are asymmetrical R_{OA} does not have the same value as R_{OB}.

Iterative impedance is the term used for the impedance measured at one port of a two-port network when the other port is terminated with an impedance of the same value. For example, the impedance looking into port 1 of Figure 41.22(a) is, say, 500 Ω when port 2 is terminated in 500 Ω and the impedance looking into port 2 of Figure 41.22(b) is, say, 600 Ω when port 1 is terminated in 600 Ω. (In symmetric T- and π-sections the two iterative impedances are equal, this value being the characteristic impedance of the section.)

An **image impedance** is defined as the impedance which, when connected to the terminals of a network, equals the impedance presented to it at the opposite terminals. For example, the impedance looking into

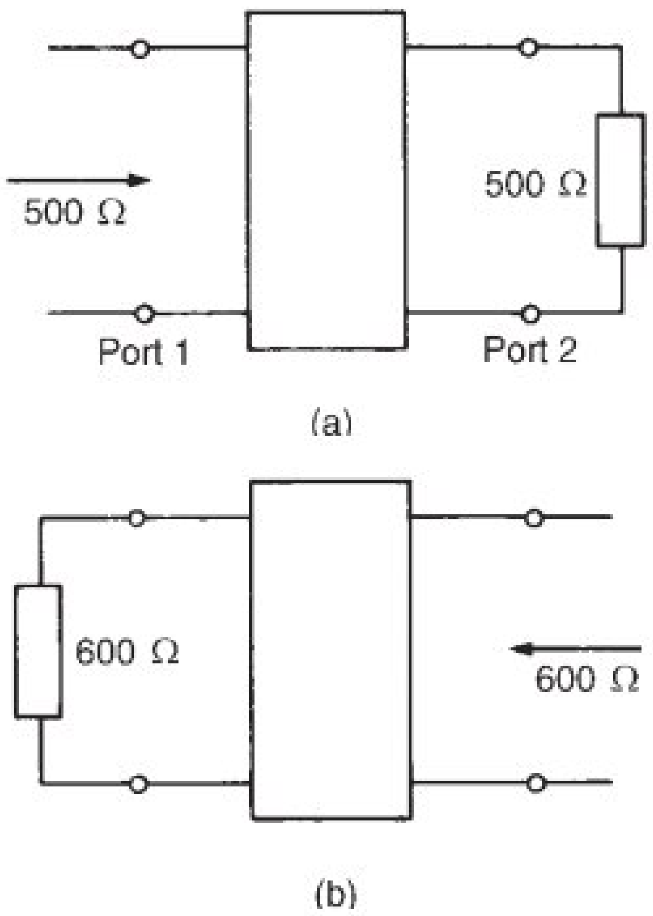


Figure 41.22

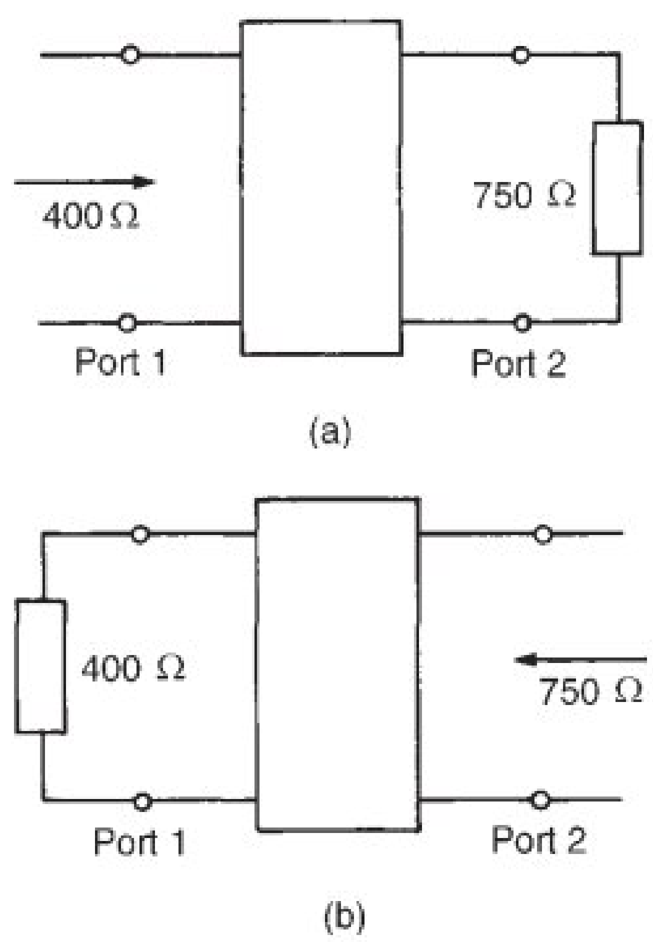


Figure 41.23

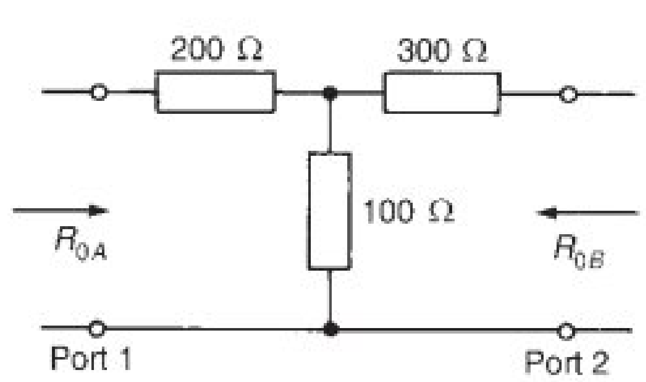


Figure 41.24

port 1 of Figure 41.23(a) is, say, 400 Ω when port 2 is terminated in, say 750 Ω, and the impedance seen looking into port 2 (Figure 41.23(b)) is 750 Ω when port 1 is terminated in 400 Ω. An asymmetrical network is correctly terminated when it is terminated in its image impedance. (If the image impedances are equal, the value is the characteristic impedance.)

The following worked problems show how the iterative and image impedances are determined for asymmetrical T- and π-sections.

> Problem 13. An asymmetrical T-section attenuator is shown in Figure 41.24. Determine for the section (a) the image impedances, and (b) the iterative impedances.

(a) The image impedance R_{OA} seen at port 1 in Figure 41.24 is given by equation (41.11): $R_{OA} = \sqrt{(R_{OC})(R_{SC})}$, where R_{OC} and R_{SC} refer to port 2 being respectively open-circuited and short-circuited.

$$R_{OC} = 200 + 100 = 300\ \Omega$$

and $$R_{SC} = 200 + \frac{(100)(300)}{100 + 300} = 275\ \Omega$$

Hence $$R_{OA} = \sqrt{[(300)(275)]} = \mathbf{287.2\ \Omega}$$

Similarly, $R_{OB} = \sqrt{(R_{OC})(R_{SC})}$, where R_{OC} and R_{SC} refer to port 1 being respectively open-circuited and short-circuited.

$$R_{OC} = 300 + 100 = 400\ \Omega$$

and $$R_{SC} = 300 + \frac{(200)(100)}{200 + 100} = 366.7\ \Omega$$

Hence $$R_{OB} = \sqrt{[(400)(366.7)]} = \mathbf{383\ \Omega}.$$

Thus the image impedances are 287.2 Ω and 383 Ω and are shown in the circuit of Figure 41.25.

(Checking:

$$R_{OA} = 200 + \frac{(100)(300 + 383)}{100 + 300 + 383} = 287.2\ \Omega$$

and $$R_{OB} = 300 + \frac{(100)(200 + 287.2)}{100 + 200 + 287.2} = 383\ \Omega)$$

(b) The iterative impedance at port 1 in Figure 41.26, is shown as R_1. Hence

$$R_1 = 200 + \frac{(100)(300 + R_1)}{100 + 300 + R_1} = 200 + \frac{30\,000 + 100R_1}{400 + R_1}$$

from which $$400R_1 + R_1^2 = 80\,000 + 200R_1 + 30\,000 + 100R_1$$

and $$R_1^2 + 100R_1 - 110\,000 = 0$$

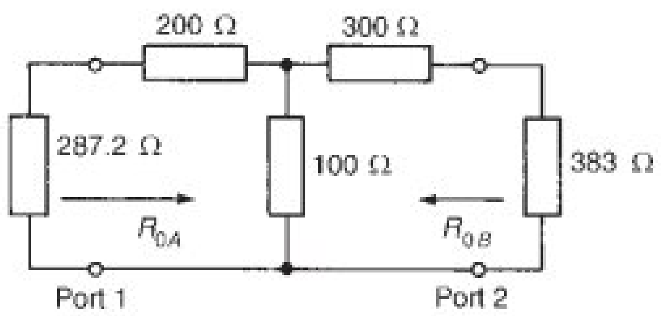


Figure 41.25

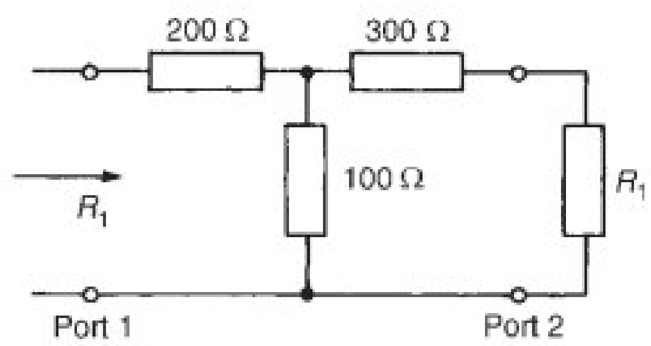


Figure 41.26

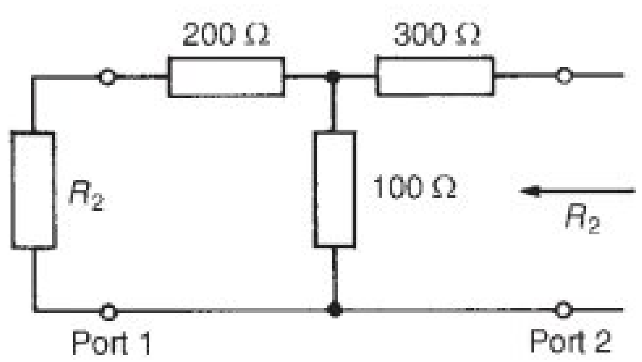


Figure 41.27

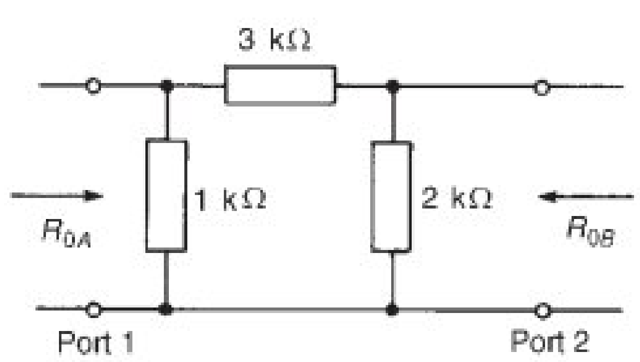


Figure 41.28

Solving by the quadratic formula gives

$$R_1 = \frac{-100 \pm \sqrt{[100^2 - (4)(1)(-110\,000)]}}{2}$$

$$= \frac{-100 \pm 670.8}{2} = \mathbf{285.4\ \Omega}$$

(neglecting the negative value).

The iterative impedance at port 2 in Figure 41.27 is shown as R_2. Hence

$$R_2 = 300 + \frac{(100)(200 + R_2)}{100 + 200 + R_2} = 300 + \frac{20\,000 + 100R_2}{300 + R_2}$$

from which $300R_2 + R_2^2 = 90\,000 + 300R_2 + 20\,000 + 100R_2$

and $R_2^2 - 100R_2 - 110\,000 = 0$

Thus $$R_2 = \frac{100 \pm \sqrt{[(-100)^2 - (4)(1)(-110\,000)]}}{2}$$

$$= \frac{100 \pm 670.8}{2} = \mathbf{385.4\ \Omega}$$

Thus the iterative impedances of the section shown in Figure 41.24 are 285.4 Ω and 385.4 Ω.

> Problem 14. An asymmetrical π-section attenuator is shown in Figure 41.28. Determine for the section (a) the image impedances, and (b) the iterative impedances.

(a) The image resistance R_{OA} seen at port 1 is given by

$$R_{OA} = \sqrt{(R_{OC})(R_{SC})},$$

where the impedance at port 1 with port 2 open-circuited,

$$R_{OC} = \frac{(1000)(5000)}{1000 + 5000} = 833\ \Omega$$

and the impedance at port 1, with port 2 short-circuited,

$$R_{SC} = \frac{(1000)(3000)}{1000 + 3000} = 750\ \Omega$$

Hence $R_{OA} = \sqrt{[(833)(750)} = \mathbf{790\ \Omega}$.

Similarly, $R_{OB} = \sqrt{(R_{OC})(R_{SC})}$, where the impedance at port 2 with port 1 open-circuited,

$$R_{OC} = \frac{(2000)(4000)}{2000 + 4000} = 1333\ \Omega$$

and the impedance at port 2 with port 1 short-circuited,

$$R_{SC} = \frac{(2000)(3000)}{2000 + 3000} = 1200\ \Omega$$

Hence $R_{OB} = \sqrt{[(1333)(1200)]} = \mathbf{1265\ \Omega}$

Thus the image impedances are 790 Ω and 1265 Ω.

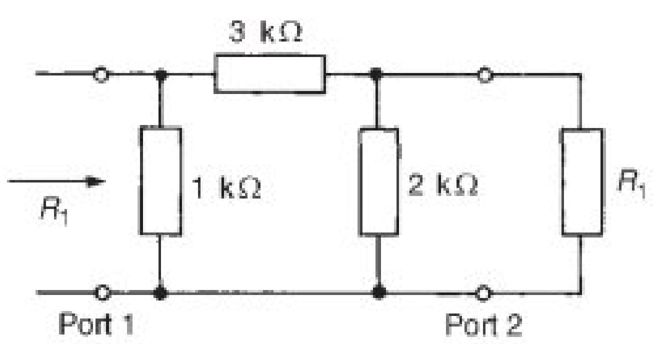


Figure 41.29

(b) The iterative impedance at port 1 in Figure 41.29 is shown as R_1. From circuit theory,

$$R_1 = \frac{1000[3000 + (2000R_1/(2000 + R_1))]}{1000 + 3000 + (2000R_1/(2000 + R_1))}$$

i.e., $$R_1 = \frac{3 \times 10^6 + (2 \times 10^6 R_1/(2000 + R_1))}{4000 + (2000R_1/(2000 + R_1))}$$

$$4000R_1 + \frac{2000R_1^2}{2000 + R_1} = 3 \times 10^6 + \frac{2 \times 10^6 R_1}{2000 + R_1}$$

$$8 \times 10^6 R_1 + 4000R_1^2 + 2000R_1^2 = 6 \times 10^9 + 3 \times 10^6 R_1 + 2 \times 10^6 R_1$$

$$6000R_1^2 + 3 \times 10^6 R_1 - 6 \times 10^9 = 0$$

$$2R_1^2 + 1000R_1 - 2 \times 10^6 = 0$$

Using the quadratic formula gives

$$R_1 = \frac{-1000 \pm \sqrt{[(1000)^2 - (4)(2)(-2 \times 10^6)]}}{4}$$

$$= \frac{-1000 \pm 4123}{4} = \mathbf{781\ \Omega}$$

(neglecting the negative value).

The iterative impedance at port 2 in Figure 41.30 is shown as R_2.

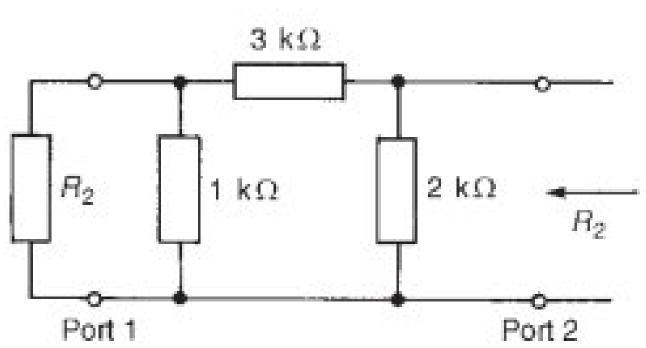


Figure 41.30

$$R_2 = \frac{2000[3000 + (1000R_2/(1000 + R_2))]}{2000 + 3000 + (1000R_2/(1000 + R_2))}$$

$$= \frac{6 \times 10^6 + (2 \times 10^6 R_2/(1000 + R_2))}{5000 + (1000R_2/(1000 + R_2))}$$

Hence

$$5000R_2 + \frac{1000R_2^2}{1000 + R_2} = 6 \times 10^6 + \frac{2 \times 10^6 R_2}{1000 + R_2}$$

$$5 \times 10^6 R_2 + 5000R_2^2 + 1000R_2^2 = 6 \times 10^9 + 6 \times 10^6 R_2 + 2 \times 10^6 R_2$$

$$6000R_2^2 - 3 \times 10^6 R_2 - 6 \times 10^9 = 0$$

$$2R_2^2 - 1000R_2 - 2 \times 10^6 = 0$$

from which

$$R_2 = \frac{1000 \pm \sqrt{[(-1000)^2 - (4)(2)(-2 \times 10^6)]}}{4}$$

$$= \frac{1000 \pm 4123}{4} = \mathbf{1281\ \Omega}$$

Thus the iterative impedances of the section shown in Figure 41.28 are 781 Ω and 1281 Ω.

Further problems on asymmetrical T—and π-sections may be found in Section 41.9, problems 19 to 21, page 787.

41.7 The L-section attenuator

A typical L-section attenuator pad is shown in Figure 41.31. Such a pad is used for matching purposes only, the design being such that the attenuation introduced is a minimum. In order to derive values for R_1 and R_2, consider the resistances seen from either end of the section.

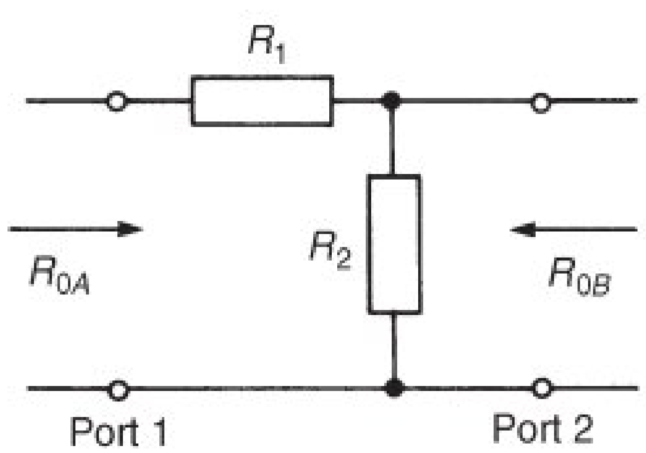


Figure 41.31 *L-section attenuator pad*

Looking in at port 1,

$$R_{OA} = R_1 + \frac{R_2 R_{OB}}{R_2 + R_{OB}}$$

from which

$$R_{OA}R_2 + R_{OA}R_{OB} = R_1R_2 + R_1R_{OB} + R_2R_{OB} \quad (41.22)$$

Looking in at port 2,

$$R_{OB} = \frac{R_2(R_1 + R_{OA})}{R_1 + R_{OA} + R_2}$$

from which

$$R_{OB}R_1 + R_{OA}R_{OB} + R_{OB}R_2 = R_1R_2 + R_2R_{OA} \quad (41.23)$$

Adding equations (41.22) and (41.23) gives

$$R_{OA}R_2 + 2R_{OA}R_{OB} + R_{OB}R_1 + R_{OB}R_2 = 2R_1R_2 + R_1R_{OB} + R_2R_{OB} + R_2R_{OA}$$

i.e., $2R_{OA}R_{OB} = 2R_1R_2$

and $$R_1 = \frac{R_{OA}R_{OB}}{R_2} \quad (41.24)$$

Substituting this expression for R_1 into equation (41.22) gives

$$R_{OA}R_2 + R_{OA}R_{OB} = \left(\frac{R_{OA}R_{OB}}{R_2}\right)R_2 + \left(\frac{R_{OA}R_{OB}}{R_2}\right)R_{OB} + R_2R_{OB}$$

i.e., $$R_{OA}R_2 + R_{OA}R_{OB} = R_{OA}R_{OB} + \frac{R_{OA}R_{OB}^2}{R_2} + R_2R_{OB}$$

from which $$R_2(R_{OA} - R_{OB}) = \frac{R_{OA}R_{OB}^2}{R_2}$$

$$R_2^2(R_{OA} - R_{OB}) = R_{OA}R_{OB}^2$$

and resistance, $$\boldsymbol{R_2 = \sqrt{\left(\frac{R_{OA}R_{OB}^2}{R_{OA} - R_{OB}}\right)}} \qquad (41.25)$$

Thus, from equation (41.24),

$$R_1 = \frac{R_{OA}R_{OB}}{\sqrt{(R_{OA}R_{OB}^2/(R_{OA} - R_{OB}))}} = \frac{R_{OA}R_{OB}}{R_{OB}\sqrt{(R_{OA}/(R_{OA} - R_{OB}))}}$$

$$= \frac{R_{OA}}{\sqrt{R_{OA}}}\sqrt{(R_{OA} - R_{OB})}$$

Hence resistance, $$\boldsymbol{R_1 = \sqrt{[R_{OA}(R_{OA} - R_{OB})]}} \qquad (41.26)$$

Figure 41.32 shows an L-section attenuator pad with its resistances expressed in terms of the input and output resistances, R_{OA} and R_{OB}.

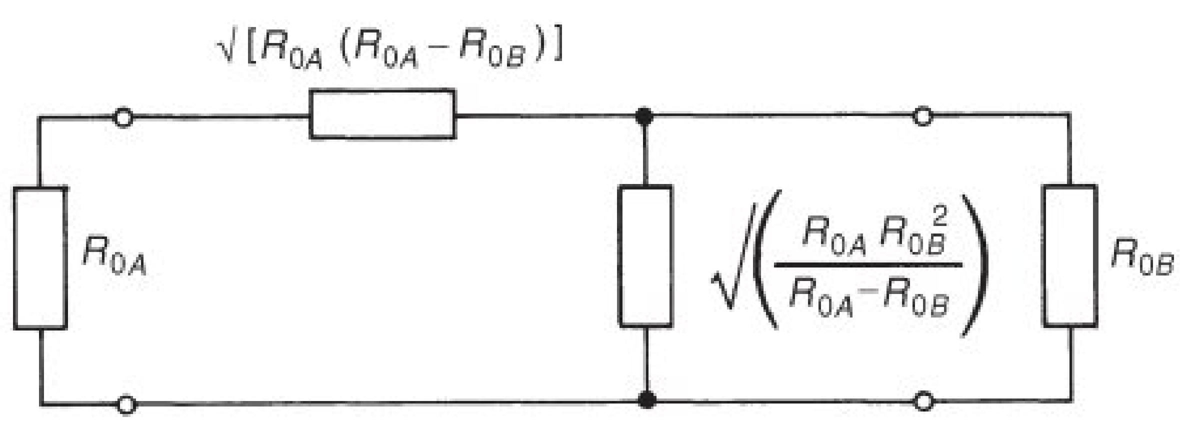


Figure 41.32

Problem 15. A generator having an internal resistance of 500 Ω is connected to a 100 Ω load via an impedance-matching resistance pad as shown in Figure 41.33. Determine (a) the values of resistance R_1 and R_2, (b) the attenuation of the pad in decibels, and (c) its insertion loss.

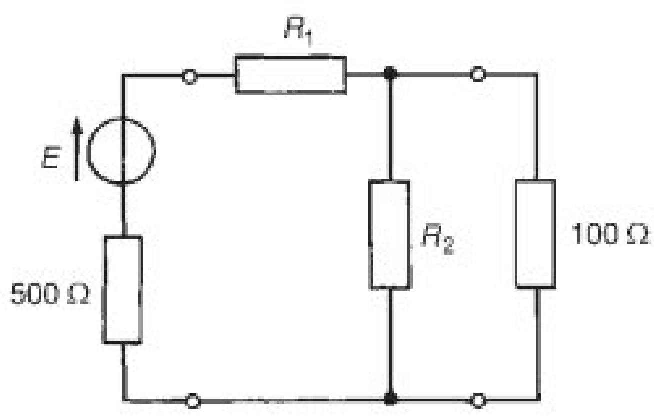


Figure 41.33

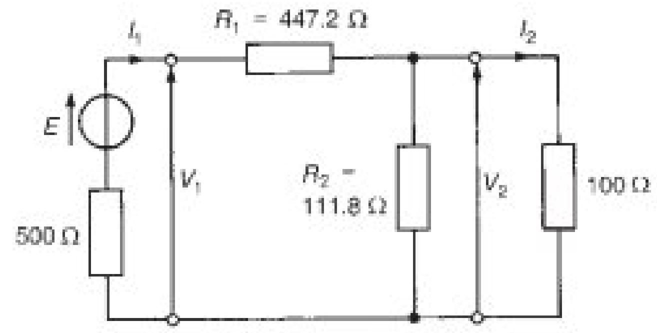


Figure 41.34

(a) From equation (41.26), $R_1 = \sqrt{[500(500-100)]} = \mathbf{447.2\ \Omega}$

From equation (41.25), $R_2 = \sqrt{\left(\dfrac{(500)(100)^2}{500-100}\right)} = \mathbf{111.8\ \Omega}$

(b) From section 41.3, the attenuation is given by $10\lg(P_1/P_2)$ dB. Note that, for an asymmetrical section such as that shown in Figure 41.33, the expression $20\ \lg(V_1/V_2)$ or $20\ \lg(I_1/I_2)$ may **not** be used for attenuation since the terminals of the pad are not matched to equal impedances. In Figure 41.34,

$$\text{current } I_1 = \frac{E}{500 + 447.2 + (111.8 \times 100/(111.8+100))} = \frac{E}{1000}$$

and current

$$I_2 = \left(\frac{111.8}{111.8+100}\right) I_1 = \left(\frac{111.8}{211.8}\right)\left(\frac{E}{1000}\right) = \frac{E}{1894.5}$$

Thus input power,

$$P_1 = I_1^2(500) = \left(\frac{E}{1000}\right)^2 (500)$$

and output power,

$$P_2 = I_2^2(100) = \left(\frac{E}{1894.5}\right)^2 (100)$$

Hence

$$\text{attenuation} = 10\lg\frac{P_1}{P_2} = 10\lg\left\{\frac{[E/(1000)]^2(500)}{[E/(1894.5]^2(100)}\right\} = 10\lg\left\{\left(\frac{1894.5}{1000}\right)^2 (5)\right\}\text{dB}$$

i.e., **attenuation = 12.54 dB**

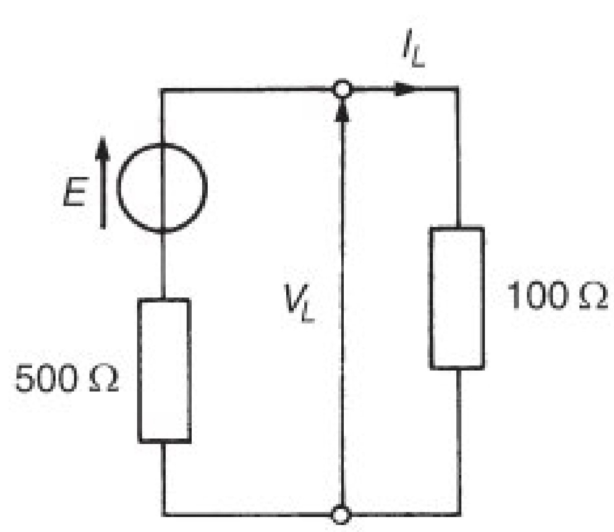


Figure 41.35

(c) Insertion loss A_L is defined as

$$\frac{\text{voltage across load when connected directly to the generator}}{\text{voltage across load when the two-port network is connected}}$$

Figure 41.35 shows the generator connected directly to the load.

Load current, $I_L = \dfrac{E}{500+100} = \dfrac{E}{600}$

and voltage, $V_L = I_L(100) = \dfrac{E}{600}(100) = \dfrac{E}{6}$

From Figure 41.34 voltage, $V_1 = E - I_1(500) = E - (E/1000)500$ from part (b)

i.e., $V_1 = 0.5$ E

voltage, $V_2 = V_1 - I_1R_1 = 0.5\text{ E} - \left(\dfrac{E}{1000}\right)(447.2) = 0.0528\text{ E}$

insertion loss, $A_L = \dfrac{V_L}{V_2} = \dfrac{E/6}{0.0528E} = \mathbf{3.157}$

In decibels, the insertion loss $= 20\lg \dfrac{V_L}{V_2}$

$= 20\lg 3.157 = \mathbf{9.99\ dB}$

Further problems on L-section attenuators may be found in Section 41.9, problems 22 and 23, page 787.

41.8 Two-port networks in cascade

Often two-port networks are connected in cascade, i.e., the output from the first network becomes the input to the second network, and so on, as shown in Figure 41.36. Thus an attenuator may consist of several cascaded sections so as to achieve a particular desired overall performance.

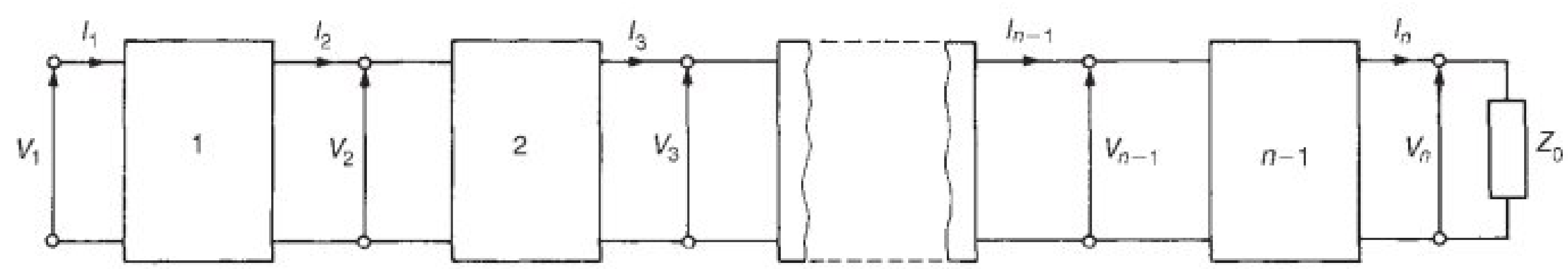


Figure 41.36 *Two-port networks connected in cascade*

If the cascade is arranged so that the impedance measured at one port and the impedance with which the other port is terminated have the same value, then each section (assuming they are symmetrical) will have the same characteristic impedance Z_0 and the last network will be terminated in Z_0. Thus each network will have a matched termination and hence the attenuation in decibels of section 1 in Figure 41.36 is given by $a_1 = 20\lg(V_1/V_2)$. Similarly, the attenuation of section 2 is given by $a_2 = 20\lg(V_2/V_3)$, and so on.

The overall attenuation is given by

$$a = 20\frac{V_1}{V_n}$$

$$= 20\lg\left(\frac{V_1}{V_2} \times \frac{V_2}{V_3} \times \frac{V_3}{V_4} \times \cdots \times \frac{V_{n-1}}{V_n}\right)$$

$$= 20\lg\frac{V_1}{V_2} + 20\lg\frac{V_2}{V_3} + \cdots + 20\lg\frac{V_{n-1}}{V_n}$$

by the laws of logarithms, i.e.,

$$\textbf{overall attenuation, } a = a_1 + a_2 + \cdots + a_{n-1} \qquad (41.27)$$

Thus the overall attenuation is the sum of the attenuations (in decibels) of the matched sections.

> Problem 16. Five identical attenuator sections are connected in cascade. The overall attenuation is 70 dB and the voltage input to the first section is 20 mV. Determine (a) the attenuation of each individual attenuation section, (b) the voltage output of the final stage, and (c) the voltage output of the third stage.

(a) From equation (41.27), the overall attenuation is equal to the sum of the attenuations of the individual sections and, since in this case each section is identical, **the attenuation of each section** $= 70/5 =$ **14 dB**.

(b) If $V_1 =$ the input voltage to the first stage and $V_0 =$ the output of the final stage, then the overall attenuation $= 20\lg(V_1/V_0)$, i.e.,

$$70 = 20\lg\left(\frac{20}{V_0}\right) \text{ where } V_0 \text{ is in millivolts}$$

$$3.5 = \lg\left(\frac{20}{V_0}\right)$$

$$10^{3.5} = \frac{20}{V_0}$$

from which

$$\textbf{output voltage of final stage, } V_0 = \frac{20}{10^{3.5}} = 6.32 \times 10^{-3}\,\text{mV}$$

$$= \mathbf{6.32\ \mu V}$$

(c) The overall attenuation of three identical stages is $3 \times 14 = 42$ dB. Hence $42 = 20\lg(V_1/V_3)$, where V_3 is the voltage output of the third stage. Thus

$$\frac{42}{20} = \lg\left(\frac{20}{V_3}\right),\ 10^{42/20} = \frac{20}{V_3}$$

from which **the voltage output of the third stage,** $V_3 = 20/10^{2.1} =$ **0.159 mV**

Problem 17. A d.c. generator has an internal resistance of 450 Ω and supplies a 450 Ω load.

(a) Design a T-network attenuator pad having a characteristic impedance of 450 Ω which, when connected between the generator and the load, will reduce the load current to $\frac{1}{8}$ of its initial value.

(b) If two such networks as designed in (a) were connected in series between the generator and the load, determine the fraction of the initial current that would now flow in the load.

(c) Determine the attenuation in decibels given by four such sections as designed in (a).

The T-network attenuator is shown in Figure 41.37 connected between the generator and the load. Since it is matching equal impedances, the network is symmetrical.

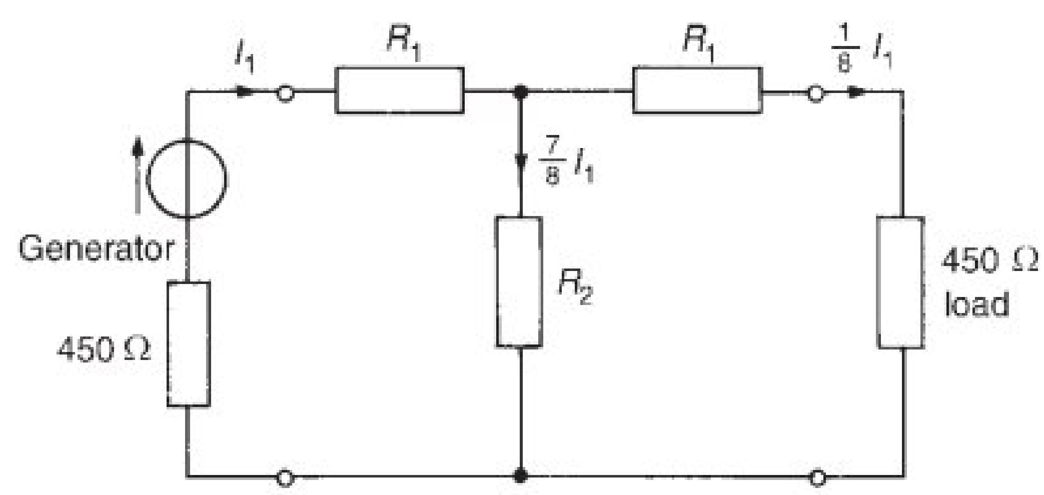


Figure 41.37

(a) Since the load current is to be reduced to $\frac{1}{8}$ of its initial value, the attenuation $N = 8$. From equation (41.13),

$$\text{resistance, } R_1 = \frac{R_0(N-1)}{(N+1)} = 450\frac{(8-1)}{(8+1)} = \mathbf{350\ \Omega}$$

and from equation (41.14),

$$\text{resistance, } R_2 = R_0\left(\frac{2N}{N^2-1}\right) = 450\left(\frac{2\times 8}{8^2-1}\right) = \mathbf{114\ \Omega}$$

(b) When two such networks are connected in series, as shown in Figure 41.38, current I_1 flows into the first stage and $\frac{1}{8}I_1$ flows out of the first stage into the second.

Again, $\frac{1}{8}$ of this current flows out of the second stage, i.e.,

$\frac{1}{8}\times\frac{1}{8}I_1$, i.e., $\frac{1}{64}$ of I_1 flows into the load.

Thus $\frac{1}{64}$ of the original current flows in the load.

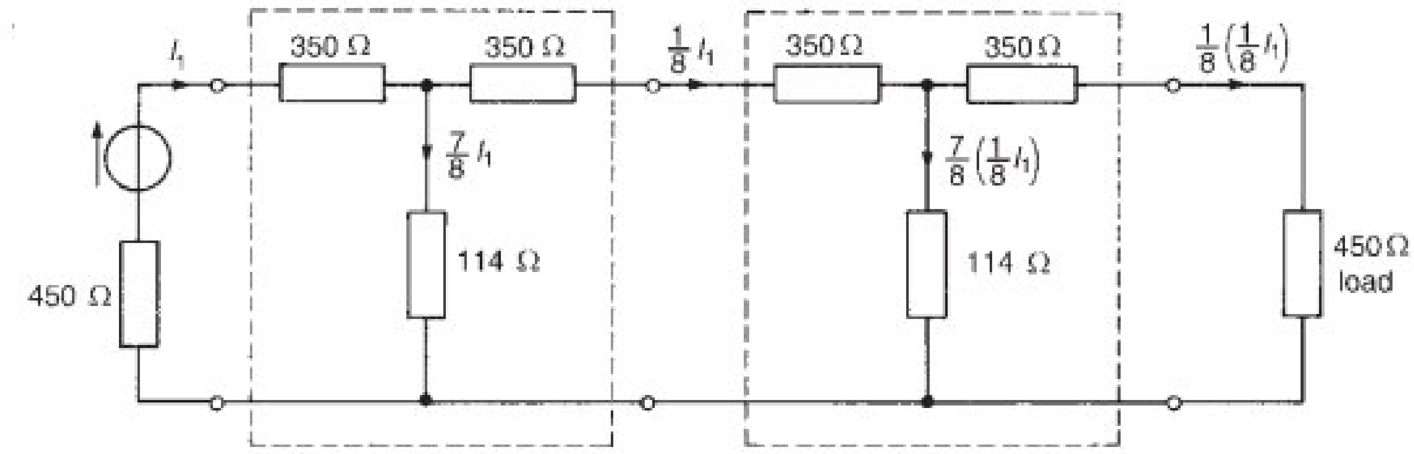


Figure 41.38

(c) The attenuation of a single stage is 8. Expressed in decibels, the attenuation is $20\lg(I_1/I_2) = 20\lg 8 = 18.06$ dB. From equation (41.27), the overall attenuation of four identical stages is given by $18.06 + 18.06 + 18.06 + 18.06$, i.e., **72.24 dB**.

Further problems on cascading two-port networks may be found in Section 41.9 following, problems 24 to 26, page 787.

41.9 Further problems on attenuators

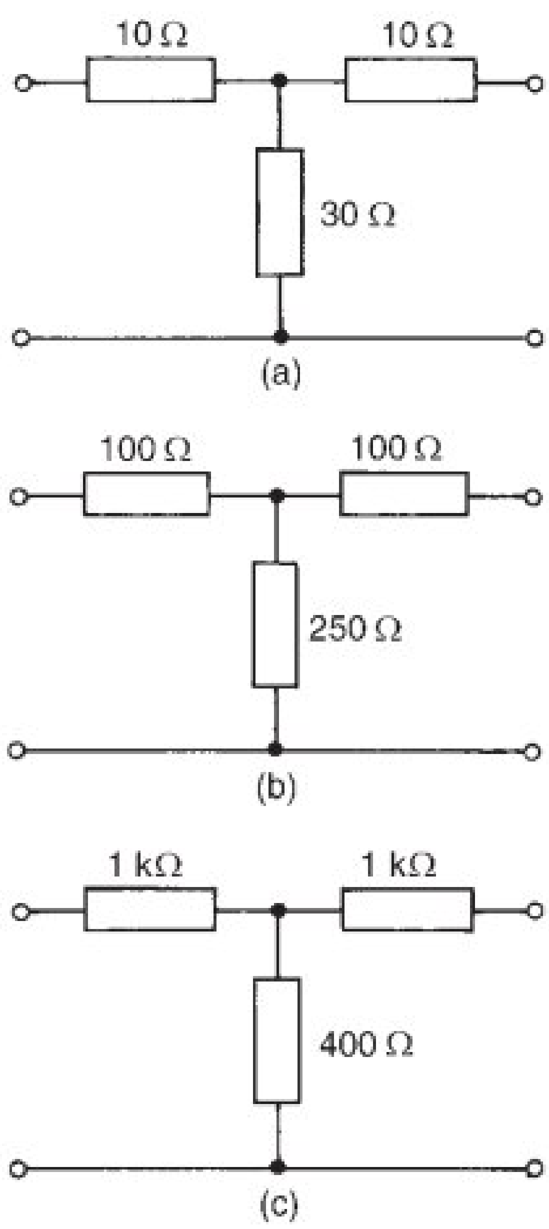


Figure 41.39

Logarithmic ratios

1 The ratio of two powers is (a) 3, (b) 10, (c) 30, (d) 10000. Determine the decibel power ratio for each.
[(a) 4.77 dB (b) 10 dB (c) 14.8 dB (d) 40 dB]

2 The ratio of two powers is (a) $\frac{1}{10}$, (b) $\frac{1}{2}$, (c) $\frac{1}{40}$, (d) $\frac{1}{1000}$. Determine the decibel power ratio for each.
[(a) −10 dB (b) −3 dB (c) −16 dB (d) −30 dB]

3 An amplifier has (a) a gain of 25 dB, (b) an attenuation of 25 dB. If the input power is 12 mW, determine the output power in each case.
[(a) 3795 mW (b) 37.9 μW]

4 7.5% of the power supplied to a cable appears at the output terminals. Determine the attenuation in decibels. [11.25 dB]

5 The current input of a system is 250 mA. If the current ratio of the system is (i) 15 dB, (ii) −8 dB, determine (a) the current output and (b) the current ratio expressed in nepers.
[(i) (a) 1.406 A (b) 1.727 Np
(ii) (a) 99.53 mA (b) − 0.921 Np]

Symmetrical T — and π-attenuators

6 Determine the characteristic impedances of the T-network attenuator sections shown in Figure 41.39.
[(a) 26.46 Ω (b) 244.9 Ω (c) 1.342 kΩ]

42 Filter networks

At the end of this chapter you should be able to:

- appreciate the purpose of a filter network
- understand basic types of filter sections, i.e., low-pass, high-pass, band-pass and band-stop filters
- understand characteristic impedance and attenuation of filter sections
- understand low and high pass ladder networks
- design a low and high pass filter section
- calculate propagation coefficient and time delay in filter sections
- understand and design '*m*-derived' filter sections
- understand and design practical composite filters

42.1 Introduction

A **filter** is a network designed to pass signals having frequencies within certain bands (called **passbands**) with little attenuation, but greatly attenuates signals within other bands (called **attenuation bands** or **stopbands**).

As explained in the previous chapter, an attenuator network pad is composed of resistances only, the attenuation resulting being constant and independant of frequency. However, a filter is frequency sensitive and is thus composed of reactive elements. Since certain frequencies are to be passed with minimal loss, ideally the inductors and capacitors need to be pure components since the presence of resistance results in some attenuation at all frequencies.

Between the pass band of a filter, where ideally the attenuation is zero, and the attenuation band, where ideally the attenuation is infinite, is the **cut-off frequency**, this being the frequency at which the attenuation changes from zero to some finite value.

A filter network containing no source of power is termed **passive**, and one containing one or more power sources is known as an **active** filter network.

The filters considered in this chapter are symmetrical unbalanced T and π sections, the reactances used being considered as ideal.

Filters are used for a variety of purposes in nearly every type of electronic communications and control equipment. The bandwidths of filters used in communications systems vary from a fraction of a hertz to many megahertz, depending on the application.

42.2 Basic types of filter sections

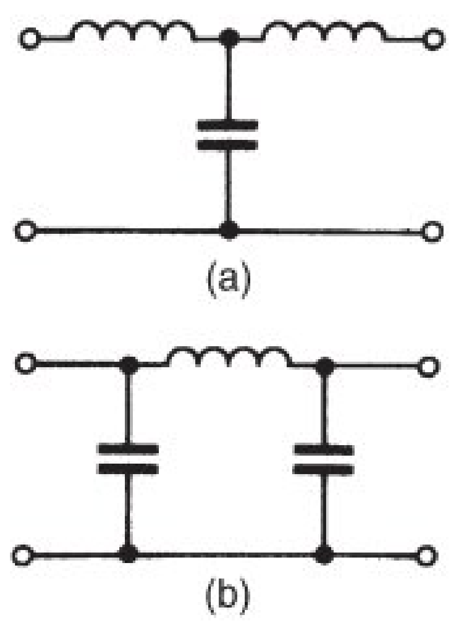


Figure 42.1

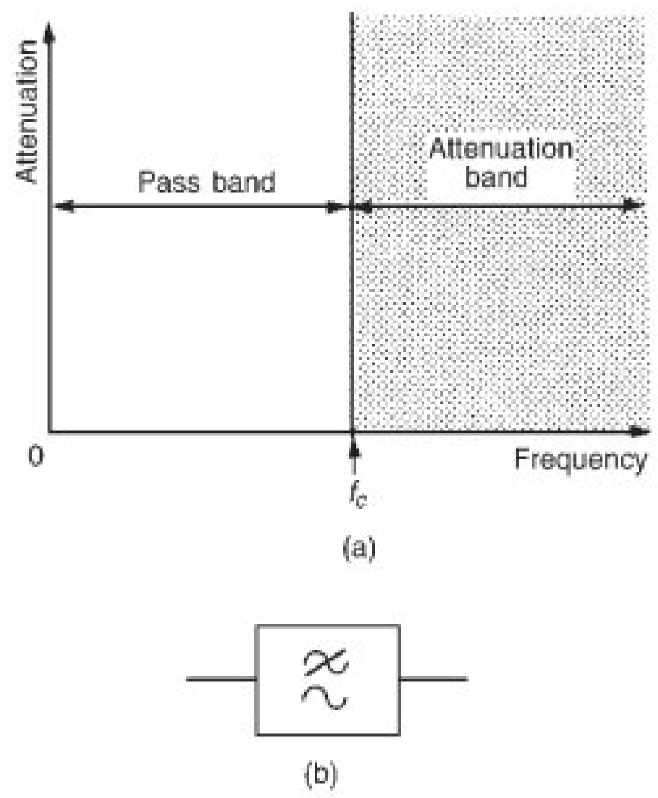


Figure 42.2

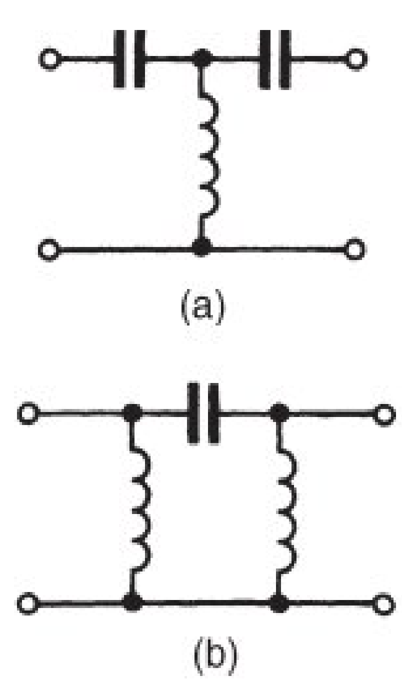


Figure 42.3

(a) Low-pass filters

Figure 42.1 shows simple unbalanced T and π section filters using series inductors and shunt capacitors. If either section is connected into a network and a continuously increasing frequency is applied, each would have a frequency-attenuation characteristic as shown in Figure 42.2(a). This is an ideal characteristic and assumes pure reactive elements. All frequencies are seen to be passed from zero up to a certain value without attenuation, this value being shown as f_c, the cut-off frequency; all values of frequency above f_c are attenuated. It is for this reason that the networks shown in Figures 42.1(a) and (b) are known as **low-pass filters**. The electrical circuit diagram symbol for a low-pass filter is shown in Figure 42.2(b).

Summarizing, a low-pass filter is one designed to pass signals at frequencies below a specified cut-off frequency.

When rectifiers are used to produce the d.c. supplies of electronic systems, a large ripple introduces undesirable noise and may even mask the effect of the signal voltage. Low-pass filters are added to smooth the output voltage waveform, this being one of the most common applications of filters in electrical circuits.

Filters are employed to isolate various sections of a complete system and thus to prevent undesired interactions. For example, the insertion of low-pass decoupling filters between each of several amplifier stages and a common power supply reduces interaction due to the common power supply impedance.

(b) High-pass filters

Figure 42.3 shows simple unbalanced T and π section filters using series capacitors and shunt inductors. If either section is connected into a network and a continuously increasing frequency is applied, each would have a frequency-attenuation characteristic as shown in Figure 42.4(a).

Once again this is an ideal characteristic assuming pure reactive elements. All frequencies below the cut-off frequency f_c are seen to be attenuated and all frequencies above f_c are passed without loss. It is for this reason that the networks shown in Figures 42.3(a) and (b) are known as **high-pass filters**. The electrical circuit-diagram symbol for a high-pass filter is shown in Figure 42.4(b).

Summarizing, a high-pass filter is one designed to pass signals at frequencies above a specified cut-off frequency.

The characteristics shown in Figures 42.2(a) and 42.4(a) are ideal in that they have assumed that there is no attenuation at all in the pass-bands and infinite attenuation in the attenuation bands. Both of these conditions are impossible to achieve in practice. Due to resistance, mainly in the inductive elements the attenuation in the pass-band will not be zero, and in a practical filter section the attenuation in the attenuation band will have a finite value. Practical characteristics for low-pass and high-pass filters are discussed in Sections 42.5 and 42.6. In addition to the resistive loss there is often an added loss due to mismatching. Ideally when a filter is inserted into a network it is matched to the impedance of that network.

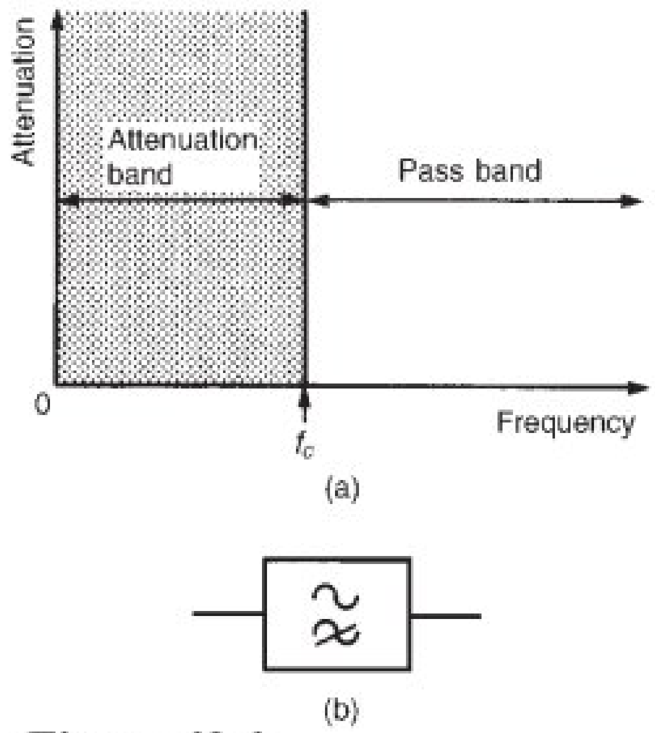


Figure 42.4

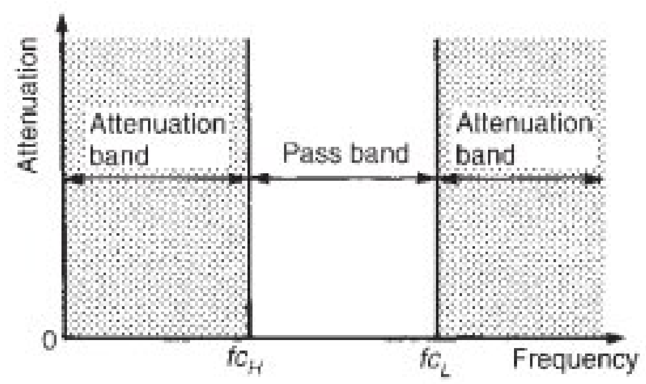


Figure 42.5

Figure 42.6

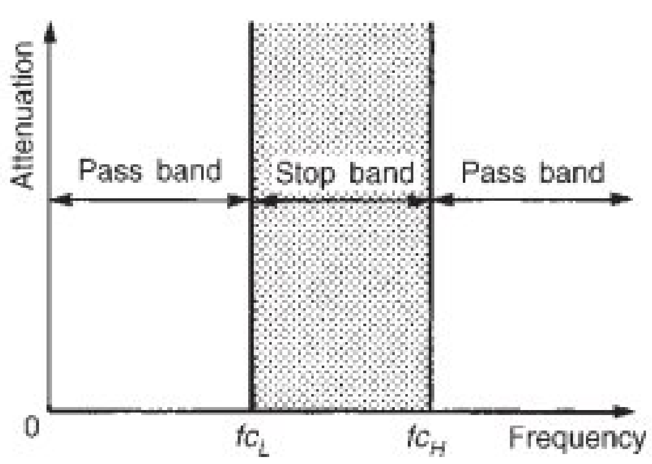


Figure 42.7

However the characteristic impedance of a filter section will vary with frequency and the termination of the section may be an impedance that does not vary with frequency in the same way. To minimize losses due to resistance and mismatching, filters are used under image impedance conditions as far as possible (see Chapter 41).

(c) Band-pass filters

A band-pass filter is one designed to pass signals with frequencies between two specified cut-off frequencies. The characteristic of an ideal band-pass filter is shown in Figure 42.5.

Such a filter may be formed by cascading a high-pass and a low-pass filter. f_{C_H} is the cut-off frequency of the high-pass filter and f_{C_L} is the cut-off frequency of the low-pass filter. As can be seen, $f_{C_L} > f_{C_H}$ for a band-pass filter, the pass-band being given by the difference between these values. The electrical circuit diagram symbol for a band-pass filter is shown in Figure 42.6.

Crystal and ceramic devices are used extensively as band-pass filters. They are common in the intermediate-frequency amplifiers of vhf radios where a precisely-defined bandwidth must be maintained for good performance.

(d) Band-stop filters

A band-stop filter is one designed to pass signals with all frequencies except those between two specified cut-off frequencies. The characteristic of an ideal band-stop filter is shown in Figure 42.7. Such a filter may be formed by connecting a high-pass and a low-pass filter in parallel. As can be seen, for a band-stop filter $f_{C_H} > f_{C_L}$, the stop-band being given by the difference between these values. The electrical circuit diagram symbol for a band-stop filter is shown in Figure 42.8.

Sometimes, as in the case of interference from 50 Hz power lines in an audio system, the exact frequency of a spurious noise signal is known. Usually such interference is from an odd harmonic of 50 Hz, for example, 250 Hz. A sharply tuned band-stop filter, designed to attenuate the 250 Hz noise signal, is used to minimize the effect of the output. A high-pass filter with cut-off frequency greater than 250 Hz would also remove the interference, but some of the lower frequency components of the audio signal would be lost as well.

42.3 The characteristic impedance and the attenuation of filter sections

Nature of the input impedance

Let a symmetrical filter section be terminated in an impedance Z_O. If the input impedance also has a value of Z_O, then Z_O is the characteristic impedance of the section.

Figure 42.9 shows a T section composed of reactive elements X_A and X_B. If the reactances are of opposite kind, then the input impedance of the section, shown as Z_O, when the output port is open or short-circuited

Figure 42.8

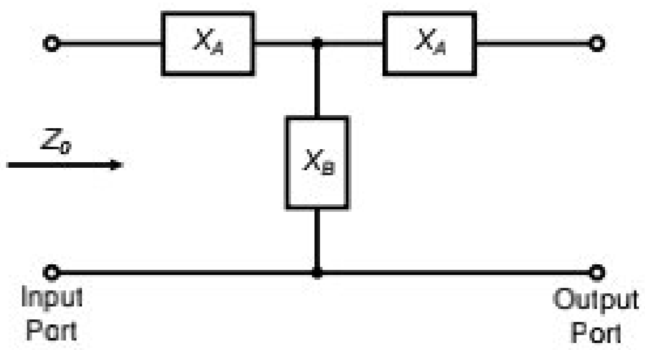


Figure 42.9

can be either inductive or capacitive depending on the frequency of the input signal.

For example, if X_A is inductive, say jX_L, and X_B is capacitive, say, $-jX_C$, then from Figure 42.9,

$$Z_{OC} = jX_L - jX_C = j(X_L - X_C)$$

and
$$Z_{SC} = jX_L + \frac{(jX_L)(-jX_C)}{(jX_L) + (-jX_C)} = jX_L + \frac{(X_L X_C)}{j(X_L - X_C)}$$

$$= jX_L - j\left(\frac{X_L X_C}{X_L - X_C}\right) = \boldsymbol{j}\left(\boldsymbol{X_L} - \frac{\boldsymbol{X_L X_C}}{\boldsymbol{X_L - X_C}}\right)$$

Since $X_L = 2\pi f L$ and $X_C = (1/2\pi f C)$ then Z_{OC} and Z_{SC} can be inductive, (i.e., positive reactance) or capacitive (i.e., negative reactance) depending on the value of frequency, f.

Let the magnitude of the reactance on open-circuit be X_{OC} and the magnitude of the reactance on short-circuit be X_{SC}. Since the filter elements are all purely reactive they may be expressed as jX_{OC} or jX_{SC}, where X_{OC} and X_{SC} are real, being positive or negative in sign. Four combinations of Z_{OC} and Z_{SC} are possible, these being:

(i) $Z_{OC} = +jX_{OC}$ and $Z_{SC} = -jX_{SC}$

(ii) $Z_{OC} = -jX_{OC}$ and $Z_{SC} = +jX_{SC}$

(iii) $Z_{OC} = +jX_{OC}$ and $Z_{SC} = +jX_{SC}$

and (iv) $Z_{OC} = -jX_{OC}$ and $Z_{SC} = -jX_{SC}$

From general circuit theory, input impedance Z_O is given by:

$$Z_O = \sqrt{(Z_{OC} Z_{SC})}$$

Taking either of combinations (i) and (ii) above gives:

$$Z_O = \sqrt{(-j^2 X_{OC} X_{SC})} = \sqrt{(X_{OC} X_{SC})},$$

which is real, thus the input impedance will be **purely resistive**.

Taking either of combinations (iii) and (iv) above gives:

$$Z_O = \sqrt{(j^2 X_{OC} X_{SC})} = +j\sqrt{(X_{OC} X_{SC})},$$

which is imaginary, thus the input impedance will be **purely reactive**.

Thus since the magnitude and nature of Z_{OC} and Z_{SC} depend upon frequency then so also will the magnitude and nature of the input impedance Z_O depend upon frequency.

Characteristic impedance

Figure 42.10 shows a low-pass T section terminated in its characteristic impedance, Z_O.

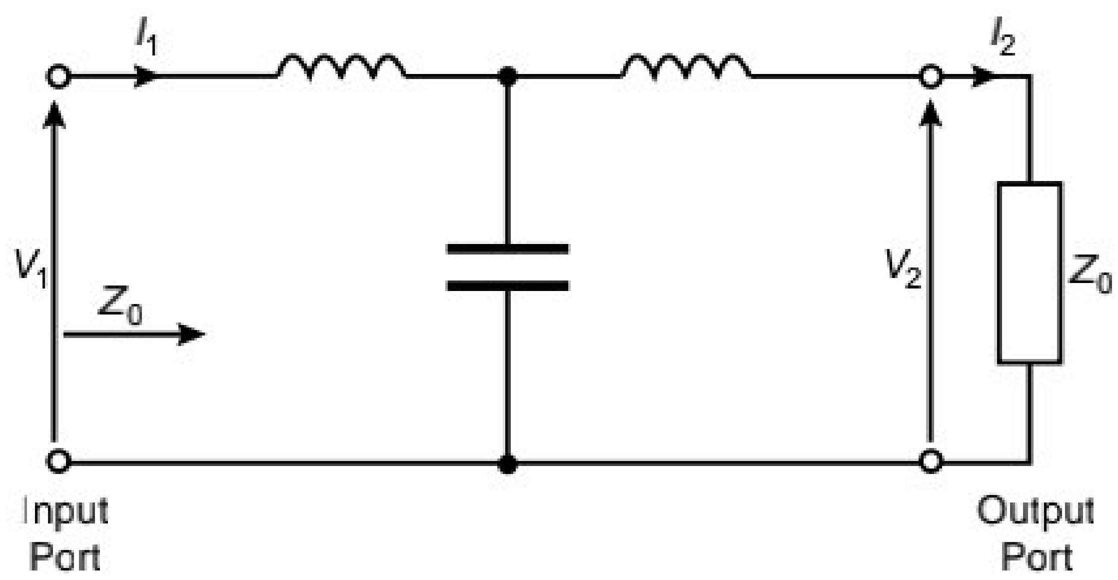


Figure 42.10

From equation (41.2), page 760, the characteristic impedance is given by $Z_O = \sqrt{(Z_{OC}Z_{SC})}$.

The following statements may be demonstrated to be true for any filter:

(a) *The attenuation is zero throughout the frequency range for which the characteristic impedance is purely resistive.*

(b) *The attenuation is finite throughout the frequency range for which the characteristic impedance is purely reactive.*

To demonstrate statement (a) above:

Let the filter shown in Figure 42.10 be operating over a range of frequencies such that Z_O is purely resistive.

From Figure 42.10, $Z_O = \dfrac{V_1}{I_1} = \dfrac{V_2}{I_2}$

Power dissipated in the output termination, $P_2 = V_2 I_2 \cos\phi_2 = V_2 I_2$ (since $\phi_2 = 0$ with a purely resistive load).

Power delivered at the input terminals,

$$P_1 = V_1 I_1 \cos\phi_1 = V_1 I_1 (\text{since } \phi_1 = 0)$$

No power is absorbed by the filter elements since they are purely reactive.

Hence $\quad P_2 = P_1$, $V_2 = V_1$ and $I_2 = I_1$.

Thus if the filter is terminated in Z_O and operating in a frequency range such that Z_O is purely resistive, then all the power delivered to the input is passed to the output and there is therefore no attenuation.

To demonstrate statement (b) above:

Let the filter be operating over a range of frequencies such that Z_O is purely reactive.

Then, from Figure 42.10, $\dfrac{V_1}{I_1} = jZ_O = \dfrac{V_2}{I_2}$.

Thus voltage and current are at 90° to each other which means that the circuit can neither accept nor deliver any active power from the source to the load ($P = VI\cos\phi = VI\cos 90° = VI(0) = 0$). There is therefore infinite attenuation, theoretically. (In practise, the attenuation is finite, for the condition $(V_1/I_1) = (V_2/I_2)$ can hold for $V_2 < V_1$ and $I_2 < I_1$, since the voltage and current are 90° out of phase.)

Statements (a) and (b) above are important because they can be applied to determine the cut-off frequency point of any filter section simply from a knowledge of the nature of Z_O. **In the pass band, Z_O is real, and in the attenuation band, Z_O is imaginary**. The cut-off frequency is therefore at the point on the frequency scale at which Z_O changes from a real quantity to an imaginary one, or vice versa (see Sections 42.5 and 42.6).

42.4 Ladder networks

Low-pass networks

Figure 42.11 shows a low-pass network arranged as a ladder or repetitive network. Such a network may be considered as a number of T or π sections in cascade. In Figure 42.12(a), a T section may be taken from the ladder by removing ABED, producing the low-pass filter section shown in Figure 42.13(a). The ladder has been cut in the centre of each of its inductive elements hence giving $L/2$ as the series arm elements in Figure 42.13(a).

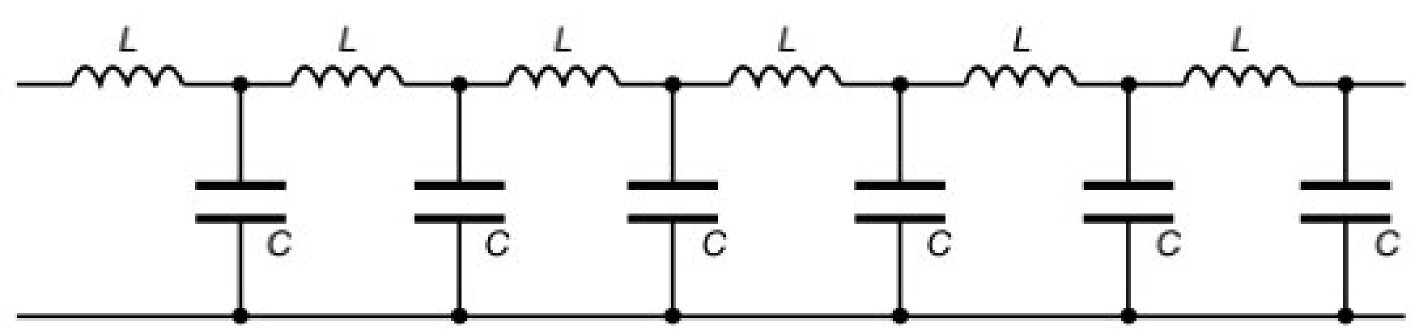


Figure 42.11

Similarly, a π section may be taken from the ladder shown in Figure 42.12(a) by removing FGJH, producing the low-pass filter section

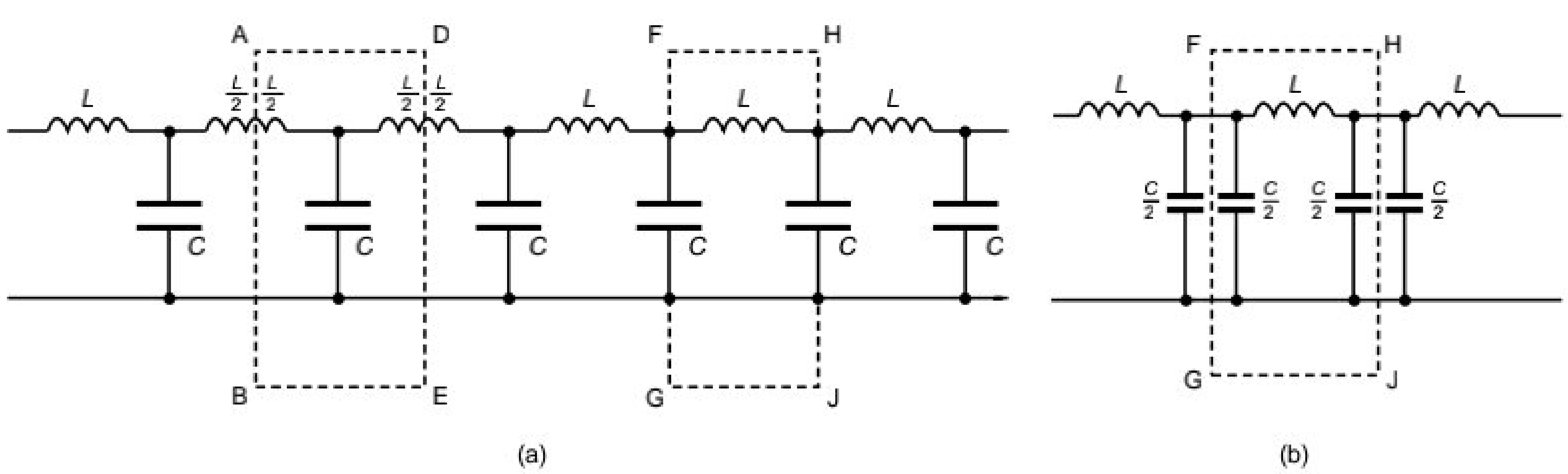


Figure 42.12

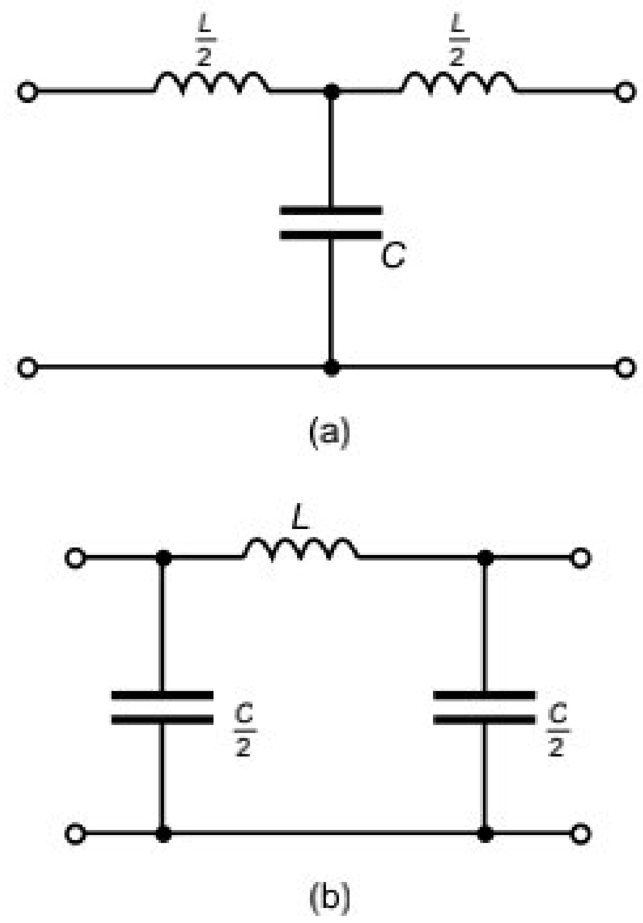


Figure 42.13

shown in Figure 42.13(b). The shunt element C in Figure 42.12(a) may be regarded as two capacitors in parallel, each of value $C/2$ as shown in the part of the ladder redrawn in Figure 42.12(b). (Note that for parallel capacitors, the total capacitance C_T is given by

$$C_T = C_1 + C_2 + \cdots. \text{ In this case } \frac{C}{2} + \frac{C}{2} = C).$$

The ladder network of Figure 42.11 can thus either be considered to be a number of the T networks shown in Figure 42.13(a) connected in cascade, or a number of the π networks shown in Figure 42.13(b) connected in cascade.

It is shown in Section 44.3, page 871, that an infinite transmission line may be reduced to a repetitive low-pass filter network.

High-pass networks

Figure 42.14 shows a high-pass network arranged as a ladder. As above, the repetitive network may be considered as a number of T or π sections in cascade.

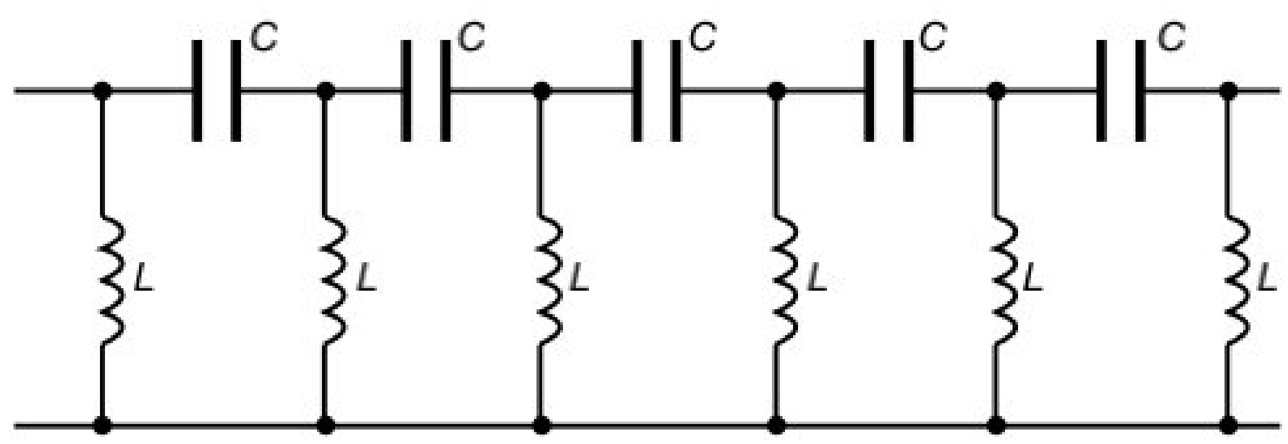


Figure 42.14

In Figure 42.15, a T section may be taken from the ladder by removing ABED, producing the high-pass filter section shown in Figure 42.16(a).

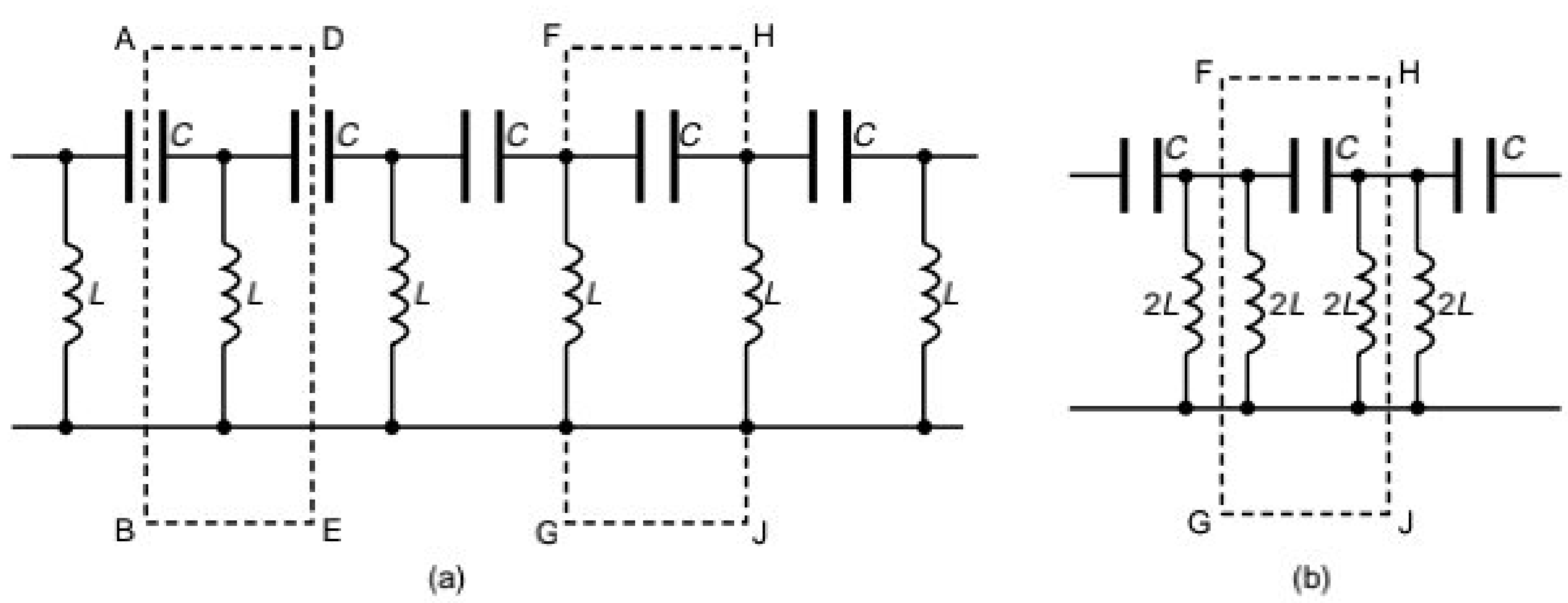
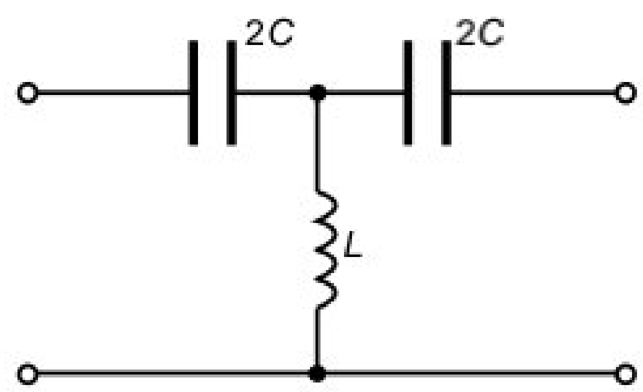
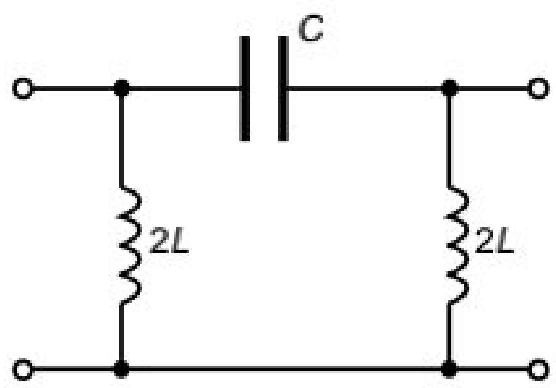


Figure 42.15



(a)



(b)

Figure 42.16

Note that the series arm elements are each $2C$. This is because two capacitors each of value $2C$ connected in series gives a total equivalent value of C, (i.e., for series capacitors, the total capacitance C_T is given by

$$\frac{1}{C_T} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots)$$

Similarly, a π section may be taken from the ladder shown in Figure 42.15 by removing FGJH, producing the high-pass filter section shown in Figure 42.16(b). The shunt element L in Figure 42.15(a) may be regarded as two inductors in parallel, each of value $2L$ as shown in the part of the ladder redrawn in Figure 42.15(b). (Note that for parallel inductance, the total inductance L_T is given by

$$\frac{1}{L_T} = \frac{1}{L_1} + \frac{1}{L_2} + \cdots. \text{ In this case, } \frac{1}{2L} + \frac{1}{2L} = \frac{1}{L}.)$$

The ladder network of Figure 42.14 can thus be considered to be either a number of T networks shown in Figure 42.16(a) connected in cascade, or a number of the π networks shown in Figure 42.16(b) connected in cascade.

42.5 Low-pass filter sections

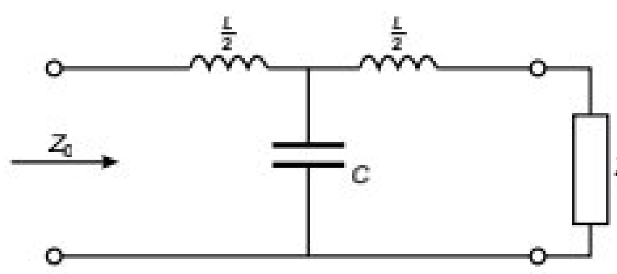


Figure 42.17

(a) The cut-off frequency

From equation (41.1), the characteristic impedance Z_0 for a symmetrical T network is given by: $Z_0 = \sqrt{(Z_A^2 + 2Z_AZ_B)}$. Applying this to the low-pass T section shown in Figure 42.17,

$$Z_A = \frac{j\omega L}{2} \text{ and } Z_B = \frac{1}{j\omega C}$$

Thus $$Z_0 = \sqrt{\left[\frac{j^2\omega^2L^2}{4} + 2\left(\frac{j\omega L}{2}\right)\left(\frac{1}{j\omega C}\right)\right]}$$

$$= \sqrt{\left(\frac{-\omega^2L^2}{4} + \frac{L}{C}\right)}$$

i.e., $$Z_0 = \sqrt{\left(\frac{L}{C} - \frac{\omega^2L^2}{4}\right)} \qquad (42.1)$$

Z_0 will be real if $\frac{L}{C} > \frac{\omega^2L^2}{4}$

Thus attenuation will commence when $\frac{L}{C} = \frac{\omega^2L^2}{4}$

i.e., when $$\omega_c^2 = \frac{4}{LC} \qquad (42.2)$$

where $\omega_c = 2\pi f_c$ and f_c is the cut-off frequency.

Thus $$(2\pi f_c)^2 = \frac{4}{LC}$$

$$2\pi f_c = \sqrt{\left(\frac{4}{LC}\right)} = \frac{2}{\sqrt{(LC)}}$$

and $$f_c = \frac{2}{2\pi\sqrt{(LC)}} = \frac{1}{\pi\sqrt{(LC)}}$$

i.e., **the cut-off frequency,** $$\boldsymbol{f_c = \frac{1}{\pi\sqrt{(LC)}}} \qquad (42.3)$$

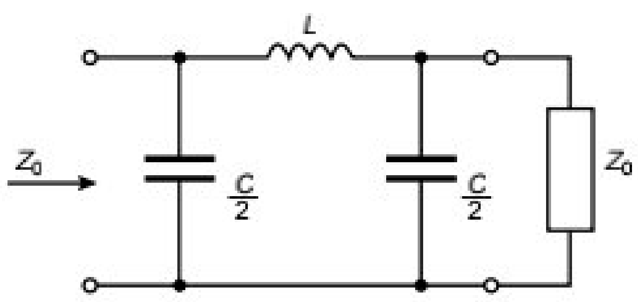


Figure 42.18

The same equation for the cut-off frequency is obtained for the low-pass π network shown in Figure 42.18 as follows:
From equation (41.3), for a symmetrical π network,

$$Z_0 = \sqrt{\left(\frac{Z_1 Z_2^2}{Z_1 + 2Z_2}\right)}$$

Applying this to Figure 42.18 $Z_1 = j\omega L$ and $Z_2 = \dfrac{1}{j\omega \frac{C}{2}} = \dfrac{2}{j\omega C}$

Thus $$Z_0 = \sqrt{\left\{\frac{(j\omega L)\left(\frac{2}{j\omega C}\right)^2}{j\omega L + 2\left(\frac{2}{j\omega C}\right)}\right\}} = \sqrt{\left\{\frac{(j\omega L)\left(\frac{4}{-\omega^2 C^2}\right)}{j\omega L - j\left(\frac{4}{\omega C}\right)}\right\}}$$

$$= \sqrt{\left\{\frac{-j\frac{4L}{\omega C^2}}{j\left(\omega L - \frac{4}{\omega C}\right)}\right\}} = \sqrt{\left\{\frac{\frac{4L}{\omega C^2}}{\frac{4}{\omega C} - \omega L}\right\}}$$

$$= \sqrt{\left\{\frac{4L}{\omega C^2\left(\frac{4}{\omega C} - \omega L\right)}\right\}} = \sqrt{\left(\frac{4L}{4C - \omega^2 LC^2}\right)}$$

i.e., $$Z_0 = \sqrt{\left(\frac{1}{\frac{C}{L} - \frac{\omega^2 C^2}{4}}\right)} \qquad (42.4)$$

Z_0 will be real if $\dfrac{C}{L} > \dfrac{\omega^2 C^2}{4}$

Thus attenuation will commence when $\frac{C}{L} = \frac{\omega^2 C^2}{4}$

i.e., when $\omega_c^2 = \frac{4}{LC}$

from which, **cut-off frequency,** $\boxed{f_c = \frac{1}{\pi\sqrt{(LC)}}}$ as in equation (42.3)).

(b) Nominal impedance

When the frequency is very low, ω is small and the term $(\omega^2 L^2/4)$ in equation (42.1) (or the term $(\omega^2 C^2/4)$ in equation (42.4)) may be neglected. The characteristic impedance then becomes equal to $\sqrt{(L/C)}$, which is purely resistive. This value of the characteristic impedance is known as the **design impedance** or the **nominal impedance** of the section and is often given the symbol R_0,

i.e., $$\boxed{R_0 = \sqrt{\frac{L}{C}}} \qquad (42.5)$$

> Problem 1. Determine the cut-off frequency and the nominal impedance of each of the low-pass filter sections shown in Figure 42.19.

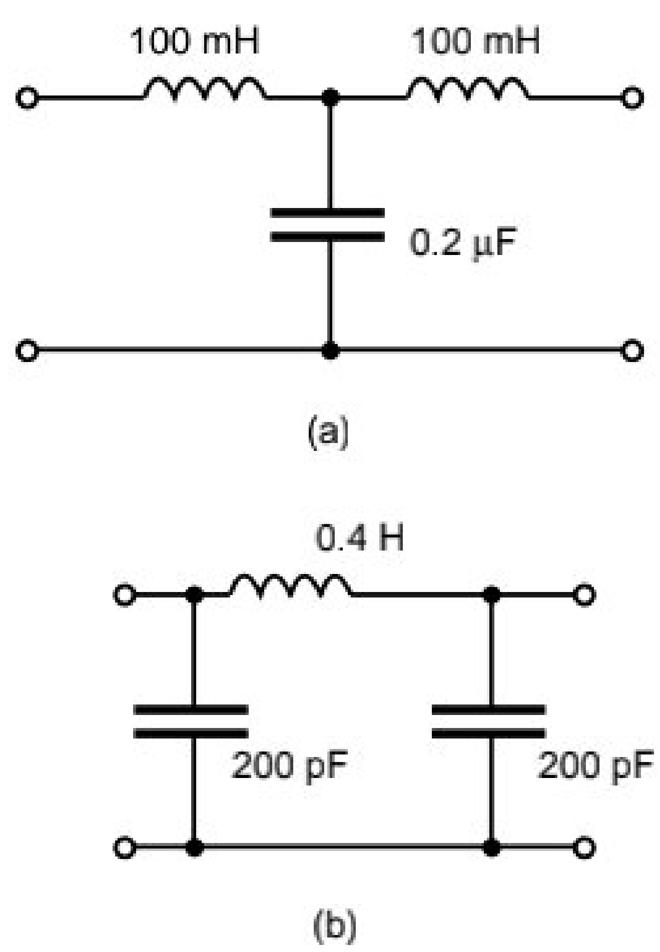


Figure 42.19

(a) Comparing Figure 42.19(a) with the low-pass T section in Figure 42.17 shows that $(L/2) = 100$ mH, i.e., inductance, $L = 200$ mH $= 0.2$ H and capacitance, $C = 0.2\ \mu\text{F} = 0.2 \times 10^{-6}$ F

From equation (42.3), cut-off frequency,

$$f_c = \frac{1}{\pi\sqrt{(LC)}} = \frac{1}{\pi\sqrt{(0.2 \times 0.2 \times 10^{-6})}} = \frac{10^3}{\pi(0.2)}$$

i.e., $f_c =$ **1592 Hz** or **1.592 kHz**

From equation (42.5), **nominal impedance,**

$$R_0 = \sqrt{\left(\frac{L}{C}\right)} = \sqrt{\left(\frac{0.2}{0.2 \times 10^{-6}}\right)} = \mathbf{1000\ \Omega} \text{ or } \mathbf{1\ k\Omega}$$

(b) Comparing Figure 42.19(b) with the low-pass π section shown in Figure 42.18 shows that $(C/2) = 200$ pF, i.e., capacitance, $C = 400$ pF $= 400 \times 10^{-12}$ F and inductance, $L = 0.4$ H,

From equation (42.3), **cut-off frequency**,

$$f_c = \frac{1}{\pi\sqrt{(LC)}} = \frac{1}{\pi\sqrt{(0.4 \times 400 \times 10^{-12})}} = \mathbf{25.16\ kHz}$$

From equation (42.5), **nominal impedance**,

$$\boldsymbol{R_0} = \sqrt{\left(\frac{L}{C}\right)} = \sqrt{\left(\frac{0.4}{400 \times 10^{-12}}\right)} = \mathbf{31.62\ k\Omega}$$

From equations (42.1) and (42.4) it is seen that the characteristic impedance Z_0 varies with ω, i.e., Z_0 varies with frequency. Thus if the nominal impedance is made to equal the load impedance into which the filter feeds then the matching deteriorates as the frequency increases from zero towards f_c. It is however convention to make the terminating impedance equal to the value of Z_0 well within the pass-band, i.e., to take the limiting value of Z_0 as the frequency approaches zero. This limit is obviously $\sqrt{(L/C)}$. This means that the filter is properly terminated at very low frequency but as the cut-off frequency is approached becomes increasingly mismatched. This is shown for a low-pass section in Figure 42.20 by curve (a). It is seen that an increasing loss is introduced into the pass band. Curve (b) shows the attenuation due to the same low-pass section being correctly terminated at all frequencies. A curve lying somewhere between curves (a) and (b) will usually result for each section if several sections are cascaded and terminated in R_0, or if a matching section is inserted between the low pass section and the load.

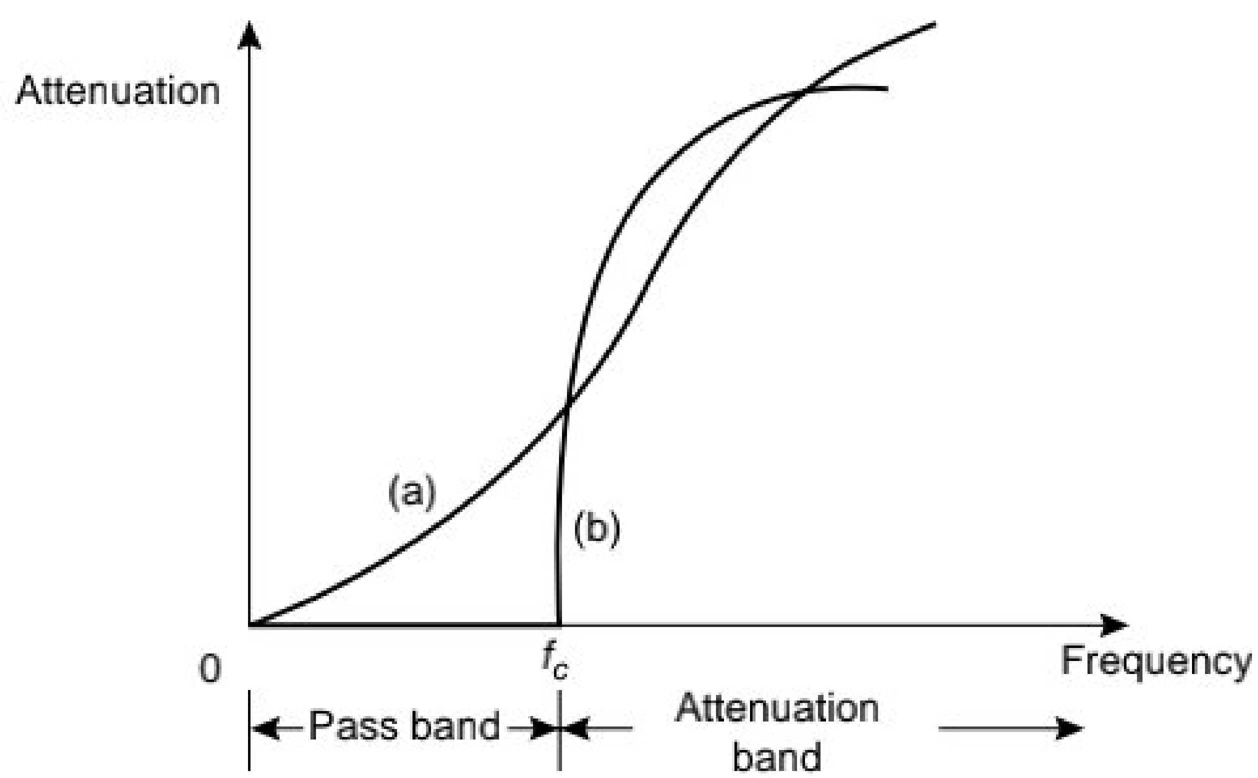


Figure 42.20

(c) To determine values of *L* and *C* given R_0 and f_c

If the values of the nominal impedance R_0 and the cut-off frequency f_c are known for a low pass T or π section it is possible to determine the values of inductance and capacitance required to form the section.

From equation (42.5), $R_0 = \sqrt{\dfrac{L}{C}} = \dfrac{\sqrt{L}}{\sqrt{C}}$ from which, $\sqrt{L} = R_0\sqrt{C}$

Substituting in equation (42.3) gives:

$$f_c = \frac{1}{\pi\sqrt{L}\sqrt{C}} = \frac{1}{\pi(R_0\sqrt{C})\sqrt{C}} = \frac{1}{\pi R_0 C}$$

from which, $$\boxed{\textbf{capacitance } C = \frac{1}{\pi R_0 f_c}} \qquad (42.6)$$

Similarly from equation (42.5), $\sqrt{C} = \dfrac{\sqrt{L}}{R_0}$

Substituting in equation (42.3) gives: $f_c = \dfrac{1}{\pi\sqrt{L}\left(\dfrac{\sqrt{L}}{R_0}\right)} = \dfrac{R_0}{\pi L}$

from which, $$\boxed{\textbf{inductance, } L = \frac{R_0}{\pi f_c}} \qquad (42.7)$$

Problem 2. A filter section is to have a characteristic impedance at zero frequency of 600 Ω and a cut-off frequency at 5 MHz. Design (a) a low-pass T section filter, and (b) a low-pass π section filter to meet these requirements.

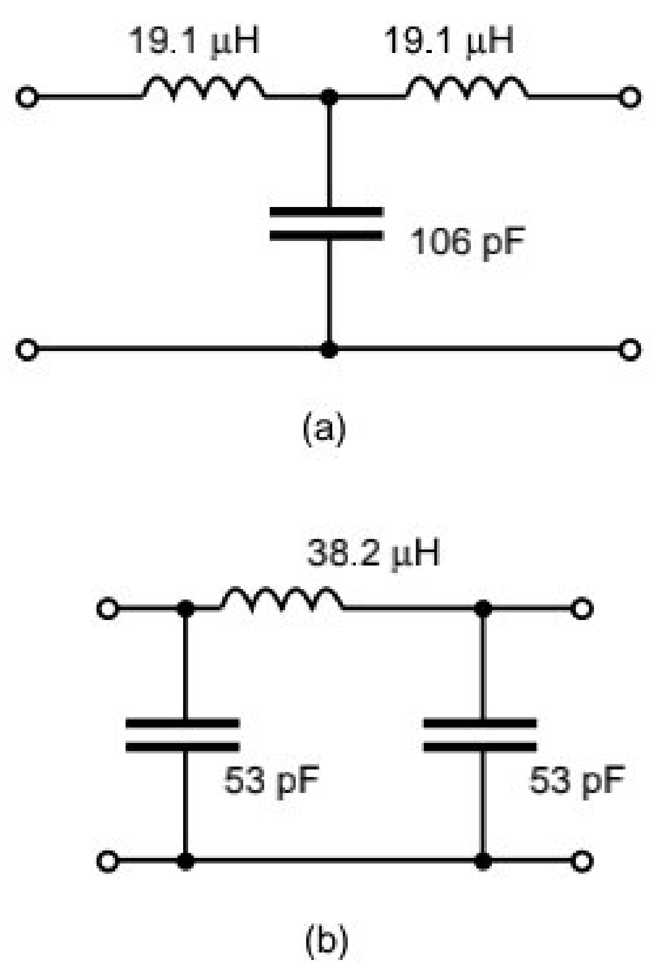


Figure 42.21

The characteristic impedance at zero frequency is the nominal impedance R_0, i.e., $R_0 = 600\ \Omega$; cut-off frequency, $f_c = 5$ MHz $= 5 \times 10^6$ Hz.

From equation (42.6),

$$\text{capacitance, } C = \frac{1}{\pi R_0 f_c} = \frac{1}{\pi(600)(5 \times 10^6)}\ \text{F} = 106\ \text{pF}$$

and from equation (42.7),

$$\text{inductance, } L = \frac{R_0}{\pi f_c} = \frac{600}{\pi(5 \times 10^6)}\ \text{H} = 38.2\ \mu\text{H}$$

(a) A low-pass T section filter is shown in Figure 42.21(a), where the series arm inductances are each L/2 (see Figure 42.17), i.e., $(38.2/2) = 19.1\ \mu$H

(b) A low-pass π section filter is shown in Figure 42.21(b), where the shunt arm capacitances are each $(C/2)$ (see Figure 42.18), i.e., $(106/2) = 53$ pF

(d) 'Constant-k' prototype low-pass filter

A ladder network is shown in Figure 42.22, the elements being expressed in terms of impedances Z_1 and Z_2. The network shown in Figure 42.22(b)

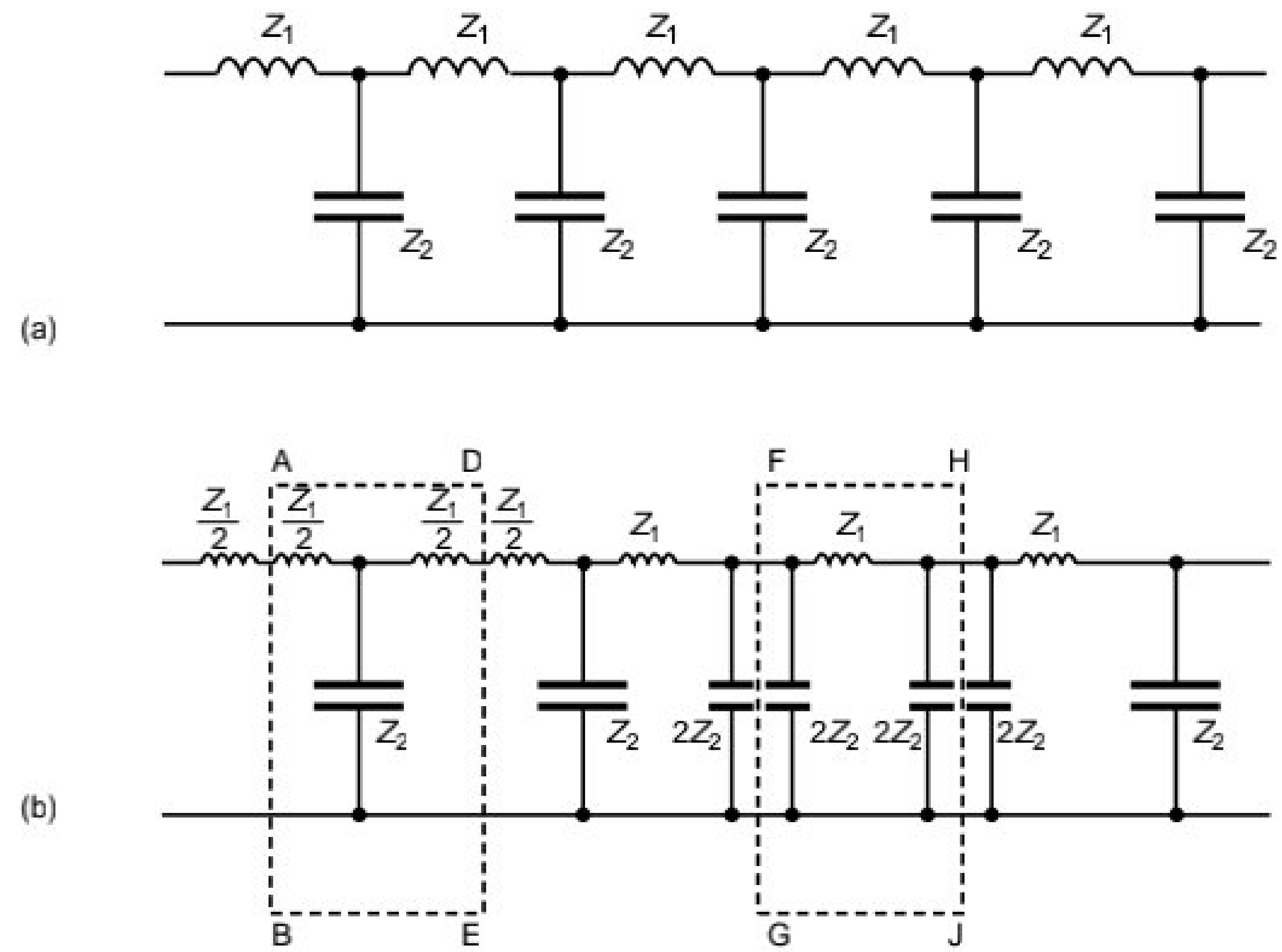


Figure 42.22

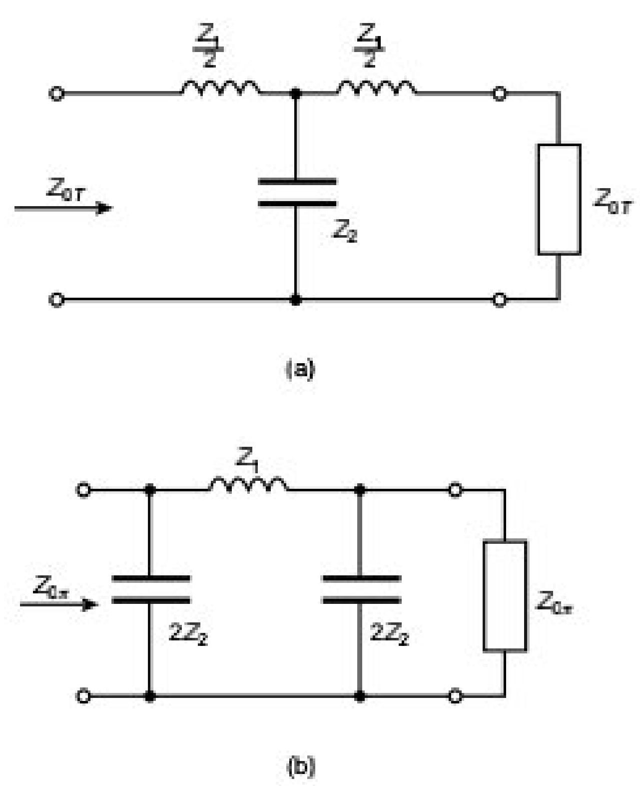


Figure 42.23

is equivalent to the network shown in Figure 42.22(a), where $(Z_1/2)$ in series with $(Z_1/2)$ equals Z_1 and $2Z_2$ in parallel with $2Z_2$ equals Z_2. Removing sections ABED and FGJH from Figure 42.22(b) gives the T section shown in Figure 42.23(a), which is terminated in its characteristic impedance Z_{OT}, and the π section shown in Figure 42.23(b), which is terminated in its characteristic impedance $Z_{0\pi}$.

From equation (41.1), page 760,

$$Z_{OT} = \sqrt{\left[\left(\frac{Z_1}{2}\right)^2 + 2\left(\frac{Z_1}{2}\right)Z_2\right]}$$

i.e., $$Z_{OT} = \sqrt{\left(\frac{Z_1^2}{4} + Z_1Z_2\right)} \qquad (42.8)$$

From equation (41.3), page 760

$$Z_{0\pi} = \sqrt{\left[\frac{(Z_1)(2Z_2)^2}{Z_1 + 2(2Z_2)}\right]} = \sqrt{\left[\frac{Z_1(Z_1)(4Z_2^2)}{Z_1(Z_1 + 4Z_2)}\right]}$$

$$= \frac{2Z_1Z_2}{\sqrt{(Z_1^2 + 4Z_1Z_2)}} = \frac{Z_1Z_2}{\sqrt{\left(\frac{Z_1^2}{4} + Z_1Z_2\right)}}$$

i.e., $Z_{0\pi} = \dfrac{Z_1Z_2}{Z_{OT}}$ from equation (42.8)

Thus $\boxed{\mathbf{Z_{0T}Z_{0\pi} = Z_1Z_2}}$ (42.9)

This is a general expression relating the characteristic impedances of T and π sections made up of equivalent series and shunt impedances.

From the low-pass sections shown in Figures 42.17 and 42.18,

$$Z_1 = j\omega L \text{ and } Z_2 = \frac{1}{j\omega C}.$$

Hence $Z_{0T}Z_{0\pi} = (j\omega L)\left(\frac{1}{j\omega C}\right) = \frac{L}{C}$

Thus, from equation (42.5), $\boxed{\mathbf{Z_{0T}Z_{0\pi} = R_0^2}}$ (42.10)

From equations (42.9) and (42.10),

$$Z_{0T}Z_{0\pi} = Z_1Z_2 = R_0^2 = \text{constant (k).}$$

A ladder network composed of reactances, the series reactances being of opposite sign to the shunt reactances (as in Figure 42.23) are called **'constant-k' filter sections**. Positive (i.e., inductive) reactance is directly proportional to frequency, and negative (i.e., capacitive) reactance is inversely proportional to frequency. Thus the product of the series and shunt reactances is independent of frequency (see equations (42.9) and (42.10)). The constancy of this product has given this type of filter its name.

From equation (42.10), it is seen that Z_{0T} and $Z_{0\pi}$ will either be both real or both imaginary together (since $j^2 = -1$). Also, when Z_{0T} changes from real to imaginary at the cut-off frequency, so will $Z_{0\pi}$. The two sections shown in Figures 42.17 and 42.18 will thus have identical cut-off frequencies and thus identical pass bands. Constant-k sections of any kind of filter are known as **prototypes**.

(e) Practical low-pass filter characteristics

From equation (42.1), the characteristic impedance Z_{0T} of a low-pass T section is given by:

$$Z_{0T} = \sqrt{\left(\frac{L}{C} - \frac{\omega^2L^2}{4}\right)}$$

Rearranging gives:

$$Z_{0T} = \sqrt{\left[\frac{L}{C}\left(1 - \frac{\omega^2LC}{4}\right)\right]} = \sqrt{\left(\frac{L}{C}\right)}\sqrt{\left(1 - \frac{\omega^2LC}{4}\right)}$$

$$= R_0\sqrt{\left(1 - \frac{\omega^2LC}{4}\right)} \text{ from equation (42.5)}$$

From equation (42.2), $\omega_c^2 = \frac{4}{LC}$, hence $Z_{0T} = R_0\sqrt{\left(1 - \frac{\omega^2}{\omega_c^2}\right)}$

i.e., $$\boxed{\boldsymbol{Z_{0T} = R_0\sqrt{\left[1 - \left(\frac{\omega}{\omega_c}\right)^2\right]}}} \qquad (42.11)$$

Also, from equation (42.10), $Z_{0\pi} = \frac{R_0^2}{Z_{0T}} = \frac{R_0^2}{R_0\sqrt{\left[1 - \left(\frac{\omega}{\omega_c}\right)^2\right]}}$

i.e., $$\boxed{\boldsymbol{Z_{0\pi} = \frac{R_0}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_c}\right)^2\right]}}}} \qquad (42.12)$$

(Alternatively, the expression for $Z_{0\pi}$ could have been obtained from equation (42.4), where

$$Z_{0\pi} = \sqrt{\left(\frac{1}{\frac{C}{L} - \frac{\omega^2C^2}{4}}\right)} = \sqrt{\left[\frac{\frac{L}{C}}{\frac{L}{C}\left(\frac{C}{L} - \frac{\omega^2C^2}{4}\right)}\right]}$$

$$= \frac{\sqrt{\frac{L}{C}}}{\sqrt{\left(1 - \frac{\omega^2LC}{4}\right)}} = \frac{R_0}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_c}\right)^2\right]}} \text{ as above}).$$

From equations (42.11) and (42.12), when $\omega = 0$ (i.e., when the frequency is zero),

$$Z_{0T} = Z_{0\pi} = R_0.$$

At the cut-off frequency, f_c, $\omega = \omega_c$

and from equation (42.11), Z_{0T} falls to zero,

and from equation (42.12), $Z_{0\pi}$ rises to infinity.

These results are shown graphically in Figure 42.24, where it is seen that Z_{0T} decreases from R_0 at zero frequency to zero at the cut-off frequency; $Z_{0\pi}$ rises from its initial value of R_0 to infinity at f_c.

(At a frequency, $f = 0.95f_c$, for example, $Z_{0\pi} = \frac{R_0}{\sqrt{(1 - 0.95^2)}} = 3.2R_0$ from equation (42.12)).

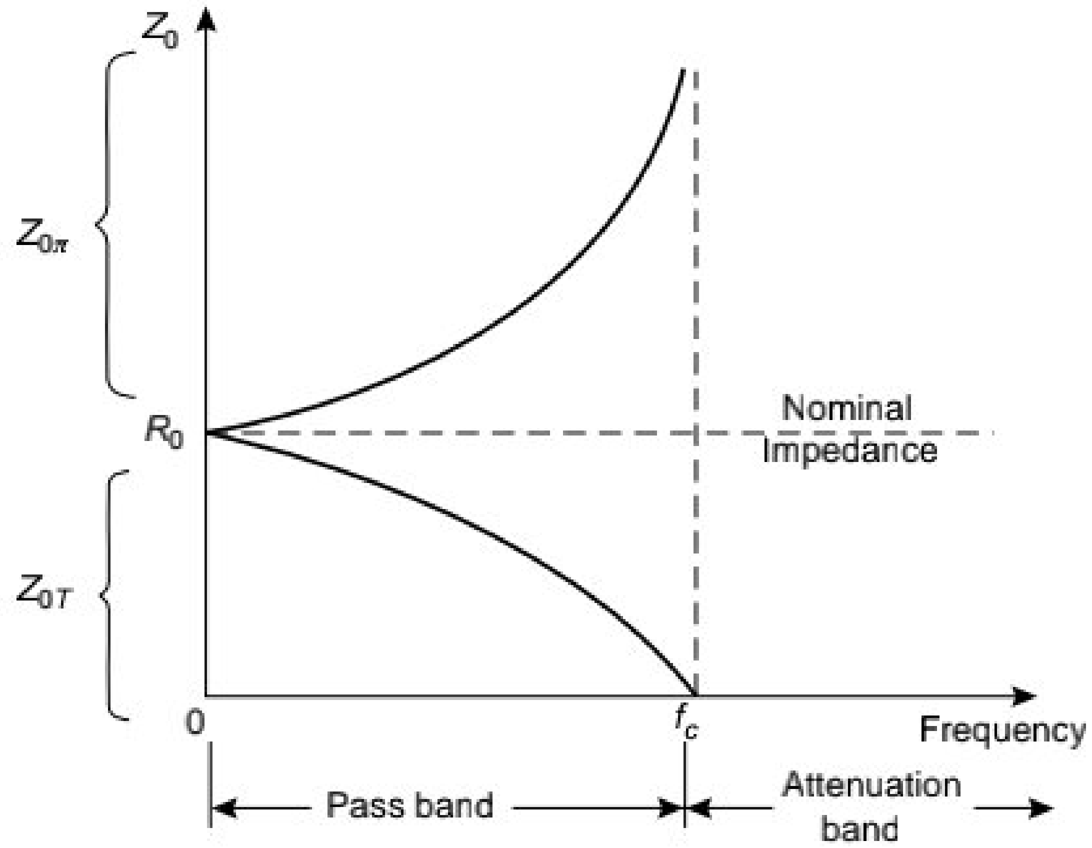


Figure 42.24

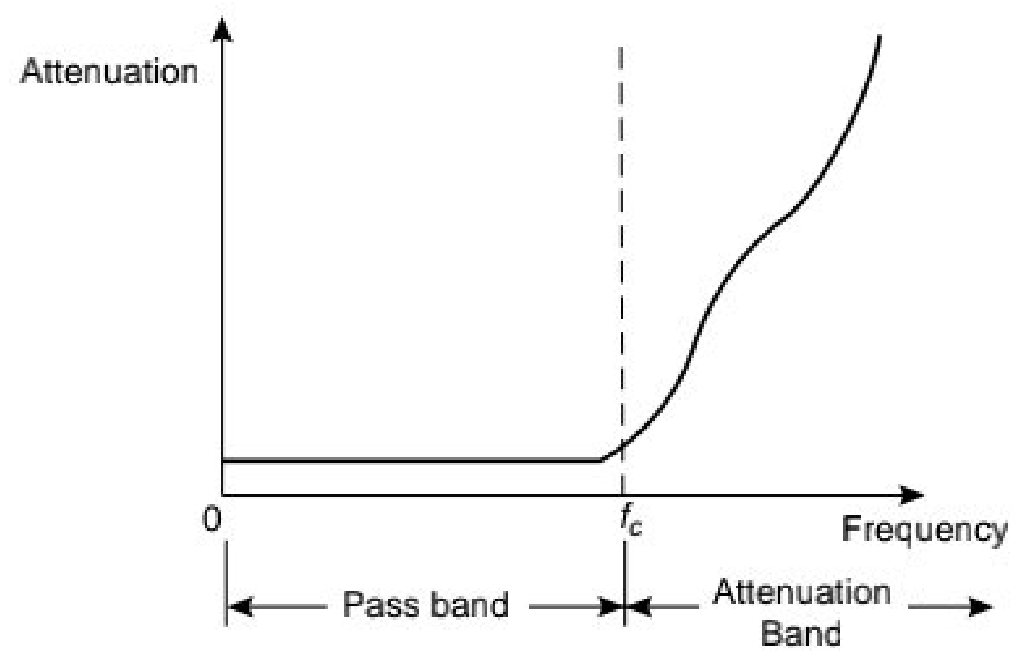


Figure 42.25

Note that since Z_0 becomes purely reactive in the attenuation band, it is not shown in this range in Figure 42.24.

Figure 42.2(a), on page 791, showed an ideal low-pass filter section characteristic. In practise, the characteristic curve of a low-pass prototype filter section looks more like that shown in Figure 42.25. The characteristic may be improved somewhat closer to the ideal by connecting two or more identical sections in cascade. This produces a much sharper cut-off characteristic, although the attenuation in the pass band is increased a little.

Problem 3. The nominal impedance of a low-pass π section filter is 500 Ω and its cut-off frequency is at 100 kHz. Determine (a) the value of the characteristic impedance of the section at a frequency of 90 kHz, and (b) the value of the characteristic impedance of the equivalent low-pass T section filter.

At zero frequency the characteristic impedance of the π and T section filters will be equal to the nominal impedance of 500 Ω.

(a) From equation (42.12), the characteristic impedance of the π section at 90 kHz is given by:

$$Z_{0\pi} = \frac{R_0}{\sqrt{\left[1-\left(\frac{\omega}{\omega_c}\right)^2\right]}} = \frac{500}{\sqrt{\left[1-\left(\frac{2\pi 90\times 10^3}{2\pi 100\times 10^3}\right)^2\right]}}$$

$$= \frac{500}{\sqrt{[1-(0.9)^2]}} = \mathbf{1147\ \Omega}$$

(b) From equation (42.11), the characteristic impedance of the T section at 90 kHz is given by:

$$Z_{0T} = R_0\sqrt{\left[1-\left(\frac{\omega}{\omega_c}\right)^2\right]} = 500\sqrt{[1-(0.9)^2]} = \mathbf{218\ \Omega}$$

(Check: From equation (42.10),

$$Z_{0T}Z_{0\pi} = (218)(1147) = 250\,000 = 500^2 = R_0^2)$$

Typical low-pass characteristics of characteristic impedance against frequency are shown in Figure 42.24.

> Problem 4. A low-pass π section filter has a nominal impedance of 600 Ω and a cut-off frequency of 2 MHz. Determine the frequency at which the characteristic impedance of the section is (a) 600 Ω (b) 1 kΩ (c) 10 kΩ

From equation (42.12), $Z_{0\pi} = \dfrac{R_0}{\sqrt{\left[1-\left(\frac{\omega}{\omega_c}\right)^2\right]}}$

(a) When $Z_{0\pi} = 600\ \Omega$ and $R_0 = 600\ \Omega$, then $\omega = 0$, i.e., **the frequency is zero**

(b) When $Z_{0\pi} = 1000\ \Omega$, $R_0 = 600\ \Omega$ and $f_c = 2\times 10^6$ Hz

then $$1000 = \frac{600}{\sqrt{\left[1-\left(\frac{2\pi f}{2\pi 2\times 10^6}\right)^2\right]}}$$

from which, $1 - \left(\frac{f}{2 \times 10^6}\right)^2 = \left(\frac{600}{1000}\right)^2 = 0.36$

and $\left(\frac{f}{2 \times 10^6}\right) = \sqrt{(1 - 0.36)} = 0.8$

Thus when $Z_{0\pi} = 1000\ \Omega$,

frequency, $\boldsymbol{f} = (0.8)(2 \times 10^6) = \mathbf{1.6\ MHz}$

(c) When $Z_{0\pi} = 10\ \mathrm{k}\Omega$, then

$$10\,000 = \frac{600}{\sqrt{\left[1 - \left(\frac{f}{2}\right)^2\right]}},$$ where frequency, f is in megahertz.

Thus $1 - \left(\frac{f}{2}\right)^2 = \left(\frac{600}{10\,000}\right)^2 = (0.06)^2$

and $\frac{f}{2} = \sqrt{[1 - (0.06)^2]} = 0.9982$

Hence when $Z_{0\pi} = 10\ \mathrm{k}\Omega$, **frequency** $\boldsymbol{f} = (2)(0.9982)$

$= \mathbf{1.996\ MHz}$

The above three results are seen to be borne out in the characteristic of $Z_{0\pi}$ against frequency shown in Figure 42.24.

Further problems on low-pass filter sections may be found in Section 42.10, problems 1 to 6, page 837.

42.6 High-pass filter sections

(a) The cut-off frequency

High-pass T and π sections are shown in Figure 42.26, (as derived in Section (42.4)), each being terminated in their characteristic impedance.

From equation (41.1), page 760, the characteristic impedance of a T section is given by:

$$Z_{0T} = \sqrt{(Z_A^2 + 2Z_A Z_B)}$$

From Figure 42.26(a), $Z_A = \frac{1}{j\omega 2C}$ and $Z_B = j\omega L$

Thus $$Z_{0T} = \sqrt{\left[\left(\frac{1}{j\omega 2C}\right)^2 + 2\left(\frac{1}{j\omega 2C}\right)(j\omega L)\right]}$$

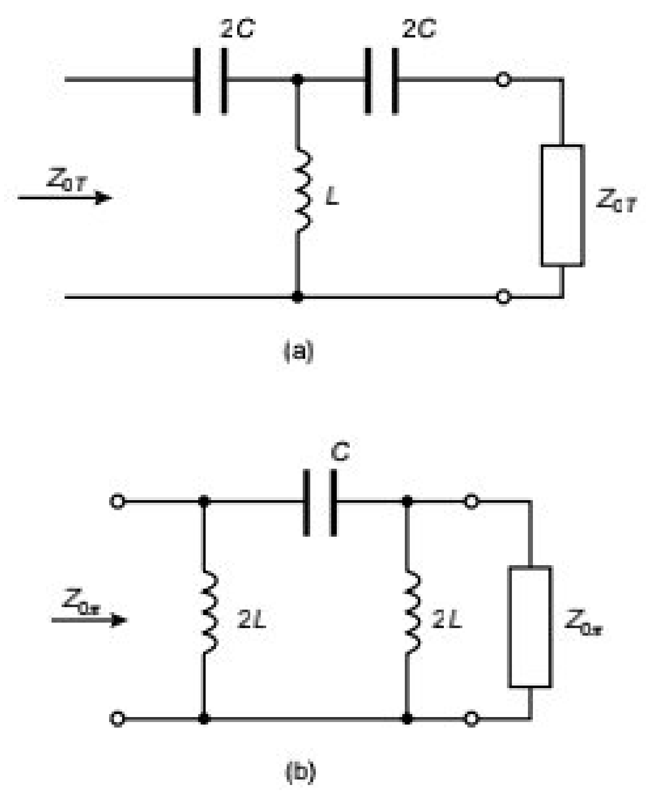


Figure 42.26

$$= \sqrt{\left[\frac{1}{-4\omega^2C^2} + \frac{L}{C}\right]}$$

i.e., $$Z_{0T} = \sqrt{\left(\frac{L}{C} - \frac{1}{4\omega^2C^2}\right)} \quad (42.13)$$

Z_{0T} will be real when $\dfrac{L}{C} > \dfrac{1}{4\omega^2C^2}$

Thus the filter will pass all frequencies above the point

$$\text{where } \frac{L}{C} = \frac{1}{4\omega^2C^2}$$

i.e., $$\text{where } \omega_c^2 = \frac{1}{4LC} \quad (42.14)$$

where $\omega_c = 2\pi f_c$, and f_c is the cut-off frequency.

Hence $$(2\pi f_c)^2 = \frac{1}{4LC}$$

and **the cut-off frequency**, $$\boxed{\boldsymbol{f_c = \frac{1}{4\pi\sqrt{(LC)}}}} \quad (42.15)$$

The same equation for the cut-off frequency is obtained for the high-pass π network shown in Figure 42.26(b) as follows:

From equation (41.3), page 760, the characteristic impedance of a symmetrical π section is given by:

$$Z_{0\pi} = \sqrt{\left(\frac{Z_1Z_2^2}{Z_1 + 2Z_2}\right)}$$

From Figure 42.26(b), $Z_1 = \dfrac{1}{j\omega C}$ and $Z_2 = j2\omega L$

$$\text{Hence } Z_{0\pi} = \sqrt{\left\{\frac{\left(\frac{1}{j\omega C}\right)(j2\omega L)^2}{\frac{1}{j\omega C} + 2j2\omega L}\right\}}$$

$$= \sqrt{\left\{\frac{j4\frac{\omega L^2}{C}}{j\left(4\omega L - \frac{1}{\omega C}\right)}\right\}} = \sqrt{\left(\frac{\frac{4L^2}{C}}{4L - \frac{1}{\omega^2 C}}\right)}$$

i.e., $$Z_{0\pi} = \sqrt{\left(\frac{1}{\frac{C}{L} - \frac{1}{4\omega^2L^2}}\right)} \quad (42.16)$$

$Z_{0\pi}$ will be real when $\frac{C}{L} > \frac{1}{4\omega^2L^2}$ and the filter will pass all frequencies above the point where $\frac{C}{L} = \frac{1}{4\omega^2L^2}$, i.e., where $\omega_c^2 = \frac{1}{4LC}$ as above.

Thus the cut-off frequency for a high-pass π network is also given by

$$\boldsymbol{f_c = \frac{1}{4\pi\sqrt{(LC)}}} \quad \text{(as in equation (42.15))} \qquad (42.15')$$

(b) Nominal impedance

When the frequency is very high, ω is a very large value and the term $(1/4\omega^2C^2)$ in equations (42.13) and (42.16) are extremely small and may be neglected.

The characteristic impedance then becomes equal to $\sqrt{(L/C)}$, this being the nominal impedance. Thus for a high-pass filter section the nominal impedance R_0 is given by:

$$\boldsymbol{R_0 = \sqrt{\left(\frac{L}{C}\right)}} \qquad (42.17)$$

the same as for the low-pass filter sections.

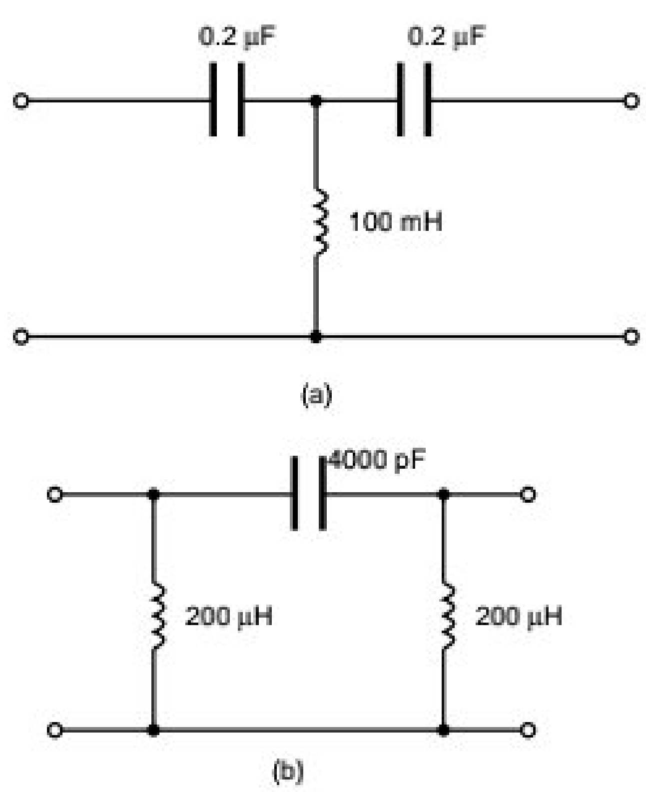


Figure 42.27

> Problem 5. Determine for each of the high-pass filter sections shown in Figure 42.27 (i) the cut-off frequency, and (ii) the nominal impedance.

(a) Comparing Figure 42.27(a) with Figure 42.26(a) shows that:

$$2C = 0.2\ \mu\text{F}, \text{ i.e., capacitance, } C = 0.1\ \mu\text{F} = 0.1 \times 10^{-6}\ \text{F}$$
$$\text{and inductance, } L = 100\ \text{mH} = 0.1\ \text{H}$$

(i) From equation (42.15),

$$\text{cut-off frequency, } f_c = \frac{1}{4\pi\sqrt{(LC)}} = \frac{1}{4\pi\sqrt{[(0.1)(0.1 \times 10^{-6}]}}$$

$$\text{i.e.,} \qquad f_c = \frac{10^3}{4\pi(0.1)} = \mathbf{796\ Hz}$$

(ii) From equation (42.17),

$$\textbf{nominal impedance, } \boldsymbol{R_0} = \sqrt{\left(\frac{L}{C}\right)} = \sqrt{\left(\frac{0.1}{0.1 \times 10^{-6}}\right)}$$
$$= \mathbf{1000\ \Omega \text{ or } 1\ k\Omega}$$

(b) Comparing Figure 42.27(b) with Figure 42.26(b) shows that:

$2L = 200\ \mu\text{H}$, i.e., inductance, $L = 100\ \mu\text{H} = 10^{-4}$ H

and capacitance $C = 4000\ \text{pF} = 4 \times 10^{-9}$ F

(i) From equation (42.15′),

$$\textbf{cut-off frequency, } f_c = \frac{1}{4\pi\sqrt{(LC)}}$$

$$= \frac{1}{4\pi\sqrt{[(10^{-4})(4 \times 10^{-9})]}} = \mathbf{126\ kHz}$$

(ii) From equation (42.17),

$$\textbf{nominal impedance, } R_0 = \sqrt{\left(\frac{L}{C}\right)} = \sqrt{\left(\frac{10^{-4}}{4 \times 10^{-9}}\right)}$$

$$= \sqrt{\left(\frac{10^5}{4}\right)} = \mathbf{158\ \Omega}$$

(c) To determine values of L and C given R_0 and f_c

If the values of the nominal impedance R_0 and the cut-off frequency f_c are known for a high-pass T or π section it is possible to determine the values of inductance L and capacitance C required to form the section.

From equation (42.17), $R_0 = \sqrt{\frac{L}{C}} = \frac{\sqrt{L}}{\sqrt{C}}$ from which, $\sqrt{L} = R_0\sqrt{C}$

Substituting in equation (42.15) gives:

$$f_c = \frac{1}{4\pi\sqrt{L}\sqrt{C}} = \frac{1}{4\pi(R_0\sqrt{C})\sqrt{C}} = \frac{1}{4\pi R_0 C}$$

from which, $$\boxed{\textbf{capacitance } C = \frac{1}{4\pi R_0 f_c}} \qquad (42.18)$$

Similarly, from equation (42.17), $\sqrt{C} = \frac{\sqrt{L}}{R_0}$

Substituting in equation (42.15) gives: $f_c = \dfrac{1}{4\pi\sqrt{L}\left(\dfrac{\sqrt{L}}{R_0}\right)} = \dfrac{R_0}{4\pi L}$

from which, $$\boxed{\textbf{inductance, } L = \frac{R_0}{4\pi f_c}} \qquad (42.19)$$

> Problem 6. A filter is required to pass all frequencies above 25 kHz and to have a nominal impedance of 600 Ω. Design (a) a high-pass T section filter and (b) a high-pass π section filter to meet these requirements.

Cut-off frequency, $f_c = 25 \times 10^3$ Hz and nominal impedance, $R_0 = 600\ \Omega$

From equation (42.18),

$$C = \frac{1}{4\pi R_0 f_c} = \frac{1}{4\pi(600)(25 \times 10^{3})}\ \text{F} = \frac{10^{12}}{4\pi(600)(25 \times 10^3)}\ \text{pF}$$

i.e., $C = 5305$ pF or 5.305 nF

From equation (42.19), inductance,

$$L = \frac{R_0}{4\pi f_c} = \frac{600}{4\pi(25 \times 10^3)}\ \text{H} = 1.91\ \text{mH}$$

(a) A high-pass T section filter is shown in Figure 42.28(a) where the series arm capacitances are each $2C$ (see Figure 42.26(a)), i.e., $2 \times 5.305 = 10.61$ nF

(b) A high-pass π section filter is shown in Figure 42.28(b), where the shunt arm inductances are each 2 L (see Figure 42.26(b)), i.e., $2 \times 1.91 = 3.82$ mH

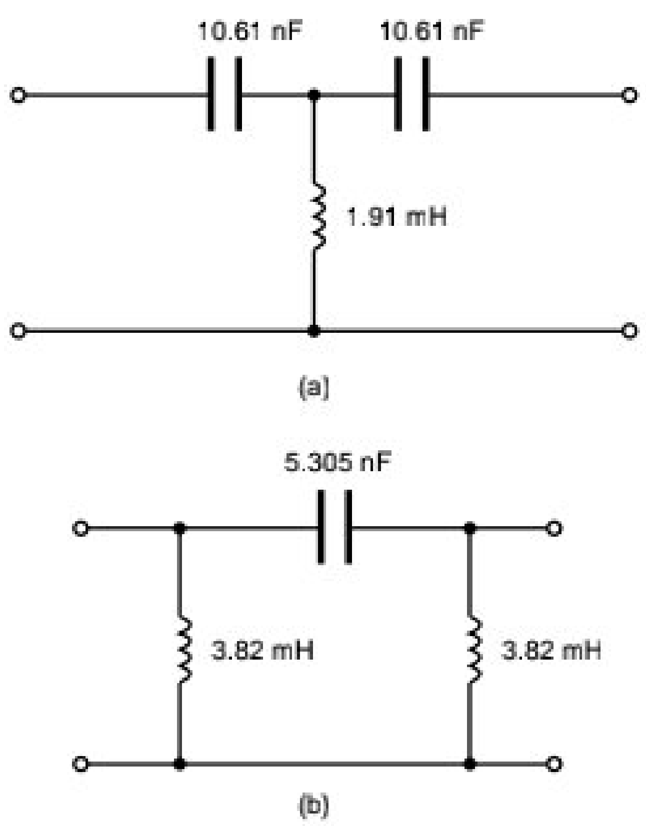


Figure 42.28

(d) 'Constant-k' prototype high-pass filter

It may be shown, in a similar way to that shown in Section 42.5(d), that for a high-pass filter section:

$$\mathbf{Z_{0T} Z_{0\pi} = Z_1 Z_2 = R_0^2}$$

where Z_1 and Z_2 are the total equivalent series and shunt arm impedances. The high-pass filter sections shown in Figure 42.26 are thus 'constant-k' prototype filter sections.

(e) Practical high-pass filter characteristics

From equation (42.13), the characteristic impedance Z_{0T} of a high-pass T section is given by:

$$Z_{0T} = \sqrt{\left(\frac{L}{C} - \frac{1}{4\omega^2 C^2}\right)}$$

Rearranging gives:

$$Z_{0T} = \sqrt{\left[\frac{L}{C}\left(1 - \frac{1}{4\omega^2 LC}\right)\right]} = \sqrt{\left(\frac{L}{C}\right)}\sqrt{\left(1 - \frac{1}{4\omega^2 LC}\right)}$$

From equation (42.14), $\omega_c^2 = \dfrac{1}{4LC}$

Thus $$\boldsymbol{Z_{0T} = R_0\sqrt{\left[1 - \left(\frac{\omega_c}{\omega}\right)^2\right]}} \qquad (42.20)$$

Also, since $Z_{0T}Z_{0\pi} = R_0^2$

then $$Z_{0\pi} = \frac{R_0^2}{Z_{0T}} = \frac{R_0^2}{R_0\sqrt{\left[1 - \left(\frac{\omega_c}{\omega}\right)^2\right]}}$$

i.e., $$\boldsymbol{Z_{0\pi} = \frac{R_0}{\sqrt{\left[1 - \left(\frac{\omega_c}{\omega}\right)^2\right]}}} \qquad (42.21)$$

From equation (42.20),

when $\omega < \omega_c$, Z_{0T} is reactive,

when $\omega = \omega_c$, Z_{0T} is zero,

and when $\omega > \omega_c$, Z_{0T} is real, eventually increasing to R_0 when ω is very large.

Similarly, from equation (42.21),

when $\omega < \omega_c$, $Z_{0\pi}$ is reactive,

when $\omega = \omega_c$, $Z_{0\pi} = \infty$ (i.e., $\dfrac{R_0}{0} = \infty$)

and when $\omega > \omega_c$, $Z_{0\pi}$ is real, eventually decreasing to R_0 when ω is very large.

Curves of Z_{0T} and $Z_{0\pi}$ against frequency are shown in Figure 42.29.

Figure 42.4(a), on page 792, showed an ideal high-pass filter section characteristic of attenuation against frequency. In practise, the characteristic curve of a high-pass prototype filter section would look more like that shown in Figure 42.30.

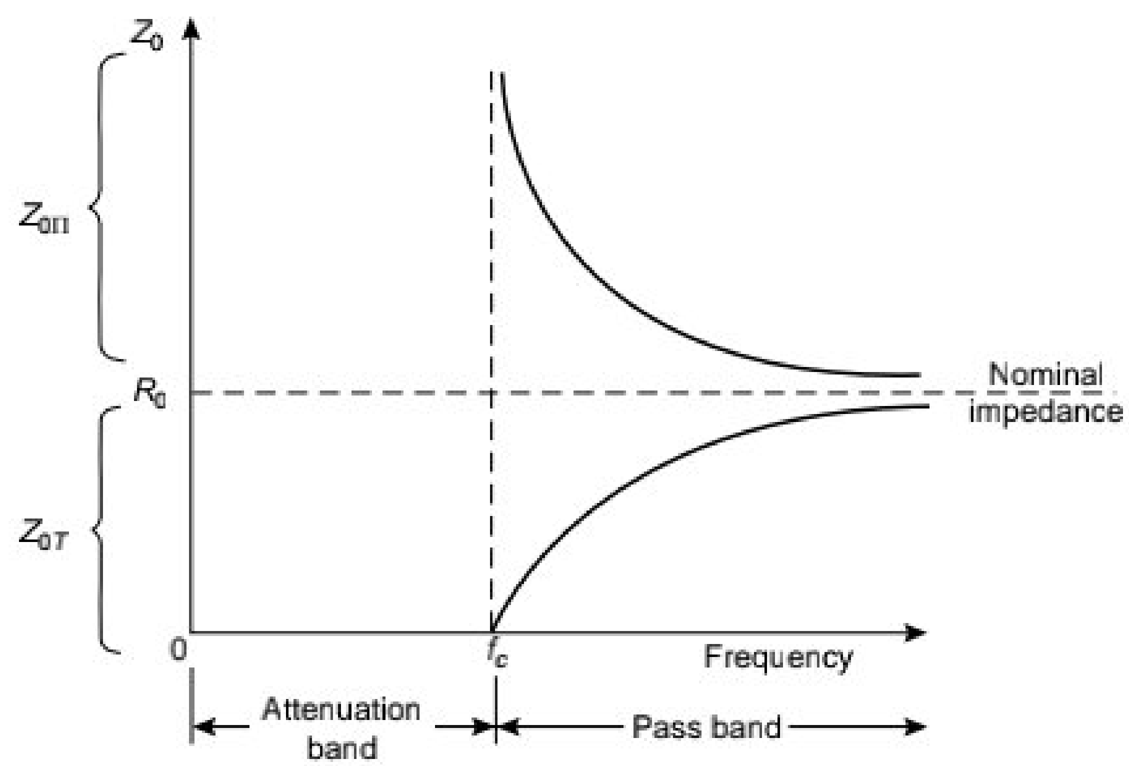


Figure 42.29

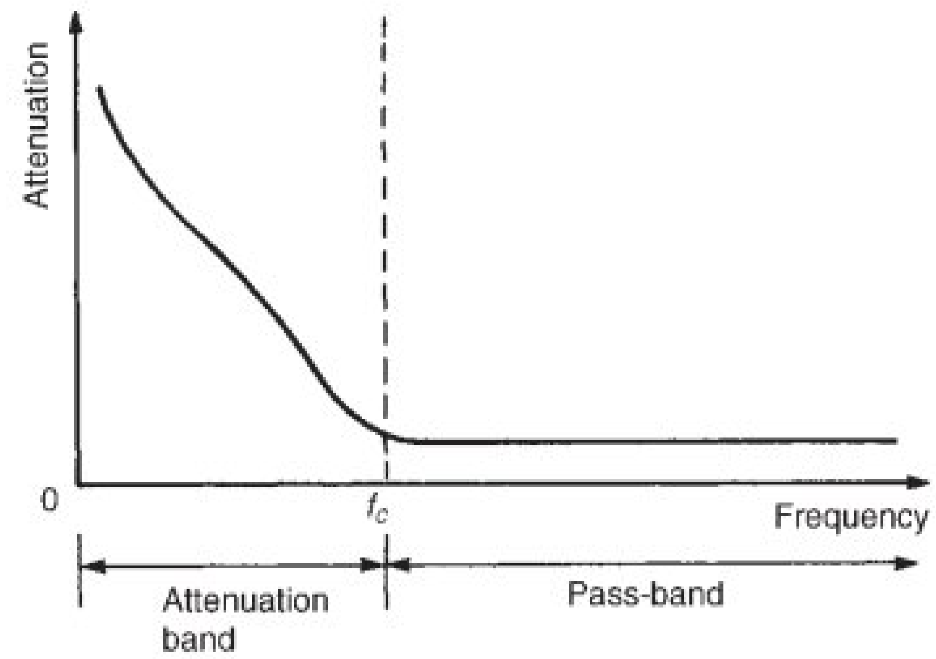


Figure 42.30

> Problem 7. A low-pass T section filter having a cut-off frequency of 15 kHz is connected in series with a high-pass T section filter having a cut-off frequency of 10 kHz. The terminating impedance of the filter is 600 Ω.
>
> (a) Determine the values of the components comprising the composite filter.
>
> (b) Sketch the expected attenuation against frequency characteristic.
>
> (c) State the name given to the type of filter described.

(a) **For the low-pass *T* section filter:** $f_{c_L} = 15\,000$ Hz
From equation (42.6),

$$\text{capacitance, } C = \frac{1}{\pi R_0 f_c} = \frac{1}{\pi(600)(15\,000)} \equiv 35.4 \text{ nF}$$

From equation (42.7),

$$\text{inductance, } L = \frac{R_0}{\pi f_c} = \frac{600}{\pi(15\,000)} \equiv 12.73 \text{ mH}$$

Thus from Figure 42.17, the series arm inductances are each $L/2$, i.e., $(12.73/2) = 6.37$ mH and the shunt arm capacitance is 35.4 nF.

For a high-pass *T* section filter: $f_{C_H} = 10\,000$ Hz
From equation (42.18),

$$\text{capacitance, } C = \frac{1}{4\pi R_0 f_c} = \frac{1}{4\pi(600)(10\,000)} \equiv 13.3 \text{ nF}$$

From equation (42.19),

$$\text{inductance, } L = \frac{R_0}{4\pi f_c} = \frac{600}{4\pi 10\,000} \equiv 4.77 \text{ mH}$$

Thus from Figure 42.26(a), the series arm capacitances are each 2 C, i.e., $2 \times 13.3 = 26.6$ nF, and the shunt arm inductance is 4.77 mH. The composite filter is shown in Figure 42.31.

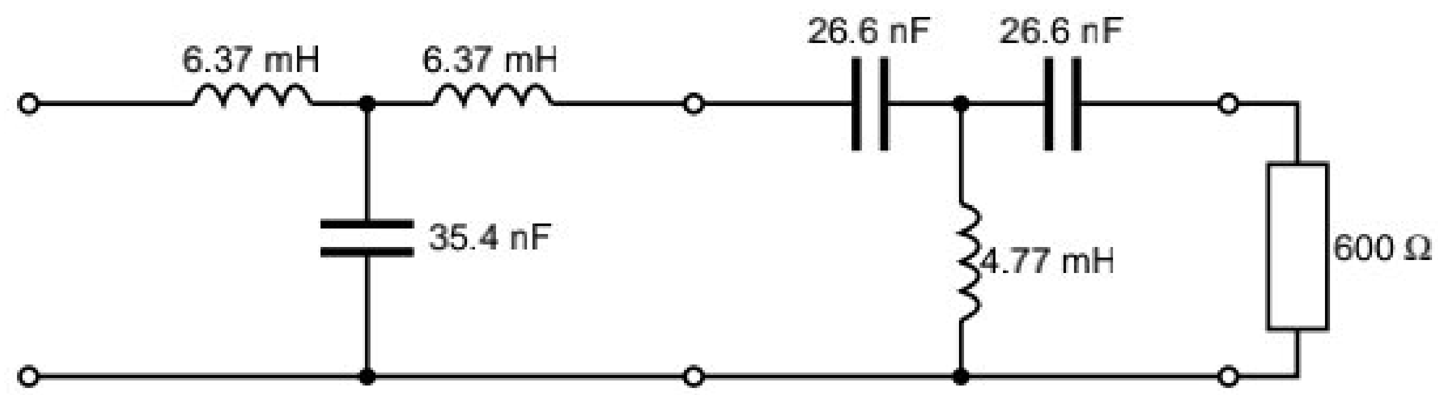


Figure 42.31

(b) A typical characteristic expected of attenuation against frequency is shown in Figure 42.32.

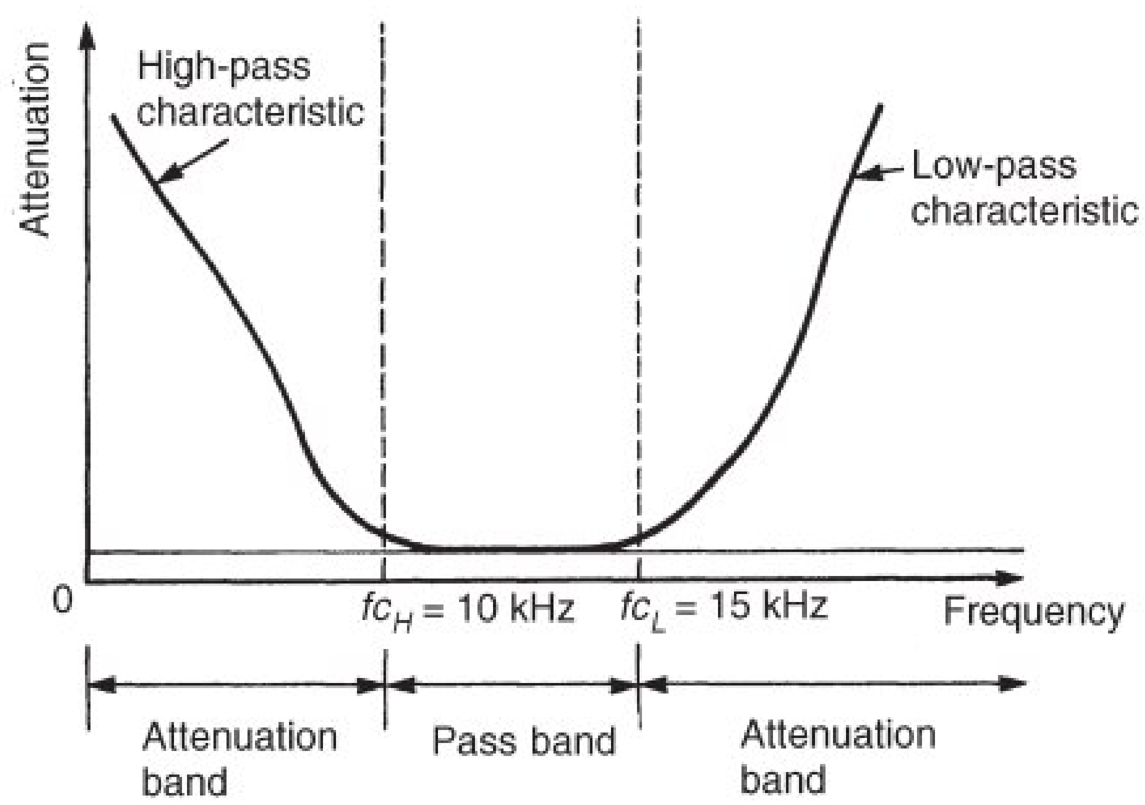


Figure 42.32

(c) The name given to the type of filter described is a **band-pass filter**. The ideal characteristic of such a filter is shown in Figure 42.5.

Problem 8. A high-pass T section filter has a cut-off frequency of 500 Hz and a nominal impedance of 600 Ω. Determine the frequency at which the characteristic impedance of the section is (a) zero, (b) 300 Ω, (c) 590 Ω.

From equation (42.20), $Z_{0T} = R_0\sqrt{\left[1 - \left(\frac{\omega_c}{\omega}\right)^2\right]}$

(a) When $Z_{0T} = 0$, then $(\omega_c/\omega) = 1$, i.e., **the frequency is 500 Hz**, the cut-off frequency.

(b) When $Z_{0T} = 300\ \Omega$, $R_0 = 600\ \Omega$ and $f_c = 500$ Hz

$$300 = 600\sqrt{\left[1 - \left(\frac{2\pi 500}{2\pi f}\right)^2\right]}$$

from which $$\left(\frac{300}{600}\right)^2 = 1 - \left(\frac{500}{f}\right)^2$$

and $$\frac{500}{f} = \sqrt{\left[1 - \left(\frac{300}{600}\right)^2\right]} = \sqrt{0.75}$$

Thus when $Z_{0T} = 300\ \Omega$, **frequency,** $\boldsymbol{f} = \frac{500}{\sqrt{0.75}} = \mathbf{577.4\ Hz}$

(c) When $Z_{0T} = 590\ \Omega$, $590 = 600\sqrt{\left[1 - \left(\frac{500}{f}\right)^2\right]}$

$$\frac{500}{f} = \sqrt{\left[1 - \left(\frac{590}{600}\right)^2\right]} = 0.1818$$

Thus when $Z_{0T} = 590\ \Omega$, **frequency,** $\boldsymbol{f} = \frac{500}{0.1818} = \mathbf{2750\ Hz}$

The above three results are seen to be borne out in the characteristic of Z_{0T} against frequency shown in Figure 42.29.

Further problems on high-pass filter sections may be found in Section 42.10, problems 7 to 12, page 837.

42.7 Propagation coefficient and time delay in filter sections

Propagation coefficient

In Figure 42.33, let A, B and C represent identical filter sections, the current ratios (I_1/I_2), (I_2/I_3) and (I_3/I_4) being equal.

Although the rate of attenuation is the same in each section (i.e., the current output of each section is one half of the current input) the amount of attenuation in each is different (section A attenuates by $\frac{1}{2}$ A, B attenuates by $\frac{1}{4}$ A and C attenuates by $\frac{1}{8}$ A). The attenuation is in fact in the

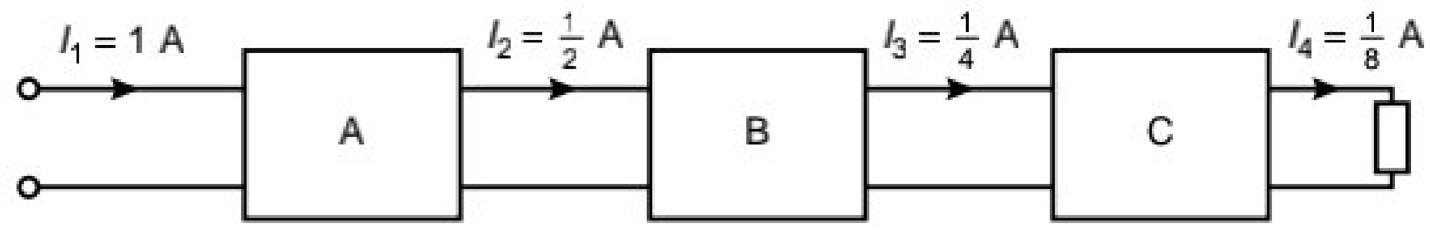


Figure 42.33

form of a logarithmic decay and

$$\frac{I_1}{I_2} = \frac{I_2}{I_3} = \frac{I_3}{I_4} = e^{\gamma} \tag{42.22}$$

where γ is called the **propagation coefficient** or the **propagation constant.**

From equation (42.22), propagation coefficient,

$$\gamma = \ln \frac{I_1}{I_2} \text{ nepers} \tag{42.23}$$

(See Section 41.3, page 761, on logarithmic units.)

Unless Sections A, B and C in Figure 42.33 are purely resistive there will be a phase change in each section. Thus the ratio of the current entering a section to that leaving it will be a phasor quantity having both modulus and argument. The propagation constant which has no units is a complex quantity given by:

$$\boxed{\gamma = \alpha + j\beta} \tag{42.24}$$

where α is called the **attenuation coefficient**, measured in nepers, and β the **phase shift coefficient**, measured in radians. β is the angle by which a current leaving a section lags behind the current entering it.

From equations (42.22) and (42.24),

$$\frac{I_1}{I_2} = e^{\gamma} = e^{\alpha + j\beta} = (e^{\alpha})(e^{j\beta})$$

Since $\quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots\ldots$

then $\quad e^{j\beta} = 1 + (j\beta) + \frac{(j\beta)^2}{2!} + \frac{(j\beta)^3}{3!} + \frac{(j\beta)^4}{4!} + \frac{(j\beta)^5}{5!} + \ldots\ldots$

$$= 1 + j\beta - \frac{\beta^2}{2!} - j\frac{\beta^3}{3!} + \frac{\beta^4}{4!} + j\frac{\beta^5}{5!} + \ldots\ldots$$

since $j^2 = -1$, $j^3 = -j$, $j^4 = +1$, and so on.

Hence $\quad e^{j\beta} = \left(1 - \frac{\beta^2}{2!} + \frac{\beta^4}{4!} - \ldots\right) + j\left(\beta - \frac{\beta^3}{3!} + \frac{\beta^5}{5!} - \ldots\ldots\right)$

$= \cos\beta + j\sin\beta$ from the power series for $\cos\beta$ and $\sin\beta$

Thus $\frac{I_1}{I_2} = e^{\alpha}e^{j\beta} = e^{\alpha}(\cos\beta + j\sin\beta) = e^{\alpha}\angle\beta$ in abbreviated polar form,

i.e., $$\frac{\boldsymbol{I_1}}{\boldsymbol{I_2}} = \boldsymbol{e^{\alpha}\angle\beta} \tag{42.25}$$

Now $$e^{\alpha} = \left|\frac{I_1}{I_2}\right|$$

from which

$$\textbf{attenuation coefficient, } \boldsymbol{\alpha} = \ln\left|\frac{\boldsymbol{I_1}}{\boldsymbol{I_2}}\right| \textbf{ nepers or 20 lg } \left|\frac{\boldsymbol{I_1}}{\boldsymbol{I_2}}\right| \textbf{ dB}$$

If in Figure 42.33 current I_2 lags current I_1 by, say, 30°, i.e., $(\pi/6)$ rad, then the propagation coefficient γ of Section A is given by:

$$\gamma = \alpha + j\beta = \ln\left|\frac{1}{\frac{1}{2}}\right| + j\frac{\pi}{6}$$

i.e., $\boldsymbol{\gamma = (0.693 + j0.524)}$

If there are n identical sections connected in cascade and terminated in their characteristic impedance, then

$$\frac{I_1}{I_{n+1}} = (e^{\gamma})^n = e^{n\gamma} = e^{n(\alpha+j\beta)} = e^{n\alpha}\angle n\beta, \ \ldots\ldots \tag{42.26}$$

where I_{n+1} is the output current of the n'th section.

Problem 9. The propagation coefficients of two filter networks are given by

(a) $\gamma = (1.25 + j0.52)$, (b) $\gamma = 1.794\angle{-39.4°}$

Determine for each (i) the attenuation coefficient, and (ii) the phase shift coefficient.

(a) If $\gamma = (1.25 + j0.52)$

then (i) the attenuation coefficient, α, is given by the real part,

i.e., $\boldsymbol{\alpha = 1.25}$ **N**

and (ii) the phase shift coefficient, β, is given by the imaginary part,

i.e., $\boldsymbol{\beta = 0.52}$ **rad**

(b) $\gamma = 1.794\angle{-39.4^\circ} = 1.794[\cos(-39.4^\circ) + j\sin(-39.4^\circ)]$

$$= (1.386 - j1.139)$$

Hence (i) the attenuation coefficient, $\boldsymbol{\alpha} = \mathbf{1.386\ N}$

and (ii) the phase shift coefficient, $\boldsymbol{\beta} = \mathbf{-1.139\ rad}$

> Problem 10. The current input to a filter section is $24\angle 10^\circ$ mA and the current output is $8\angle{-45^\circ}$ mA. Determine for the section (a) the attenuation coefficient, (b) the phase shift coefficient, and (c) the propagation coefficient. (d) If five such sections are cascaded determine the output current of the fifth stage and the overall propagation constant of the network.

Let $I_1 = 24\angle 10^\circ$ mA and $I_2 = 8\angle{-45^\circ}$ mA, then

$$\frac{I_1}{I_2} = \frac{24\angle 10^\circ}{8\angle-45^\circ} = 3\angle 55^\circ = e^{\alpha}\angle\beta \text{ from equation (42.25).}$$

(a) Hence the attenuation constant, α, is obtained from $3 = e^{\alpha}$, i.e., $\boldsymbol{\alpha} = \mathbf{\ln 3 = 1.099\ N}$

(b) The phase shift coefficient $\beta = 55^\circ \times \dfrac{\pi}{180} = \mathbf{0.960\ rad}$

(c) The propagation coefficient $\gamma = \alpha + j\beta = \mathbf{(1.099 + j0.960)}$ or $\mathbf{1.459\angle 41.14^\circ}$

(d) If I_6 is the current output of the fifth stage, then from equation (42.26),

$$\frac{I_1}{I_6} = (e^{\gamma})^n = [3\angle 55^\circ]^5 = 243\angle 275^\circ \text{ (by De Moivre's theorem)}$$

Thus the output current of the fifth stage,

$$I_6 = \frac{I_1}{243\angle 275^\circ} = \frac{24\angle 10^\circ}{243\angle 275^\circ}$$

$$= \mathbf{0.0988\angle{-265^\circ}\ mA\ or\ 98.8\angle 95^\circ\ \mu A}$$

Let the overall propagation coefficient be γ'

then $\dfrac{I_1}{I_6} = 243\angle 275^\circ = e^{\gamma'} = e^{\alpha'}\angle\beta'$

The overall attenuation coefficient $\alpha' = \ln 243 = 5.49$

and the overall phase shift coefficient $\beta' = 275^\circ \times \dfrac{\pi}{180^\circ} = 4.80$ rad

Hence the overall propagation coefficient $\boldsymbol{\gamma'} = \mathbf{(5.49 + j4.80)}$ or $\mathbf{7.29\angle 41.16^\circ}$

Problem 11. For the low-pass *T* section filter shown in Figure 42.34 determine (a) the attenuation coefficient, (b) the phase shift coefficient and (c) the propagation coefficient γ.

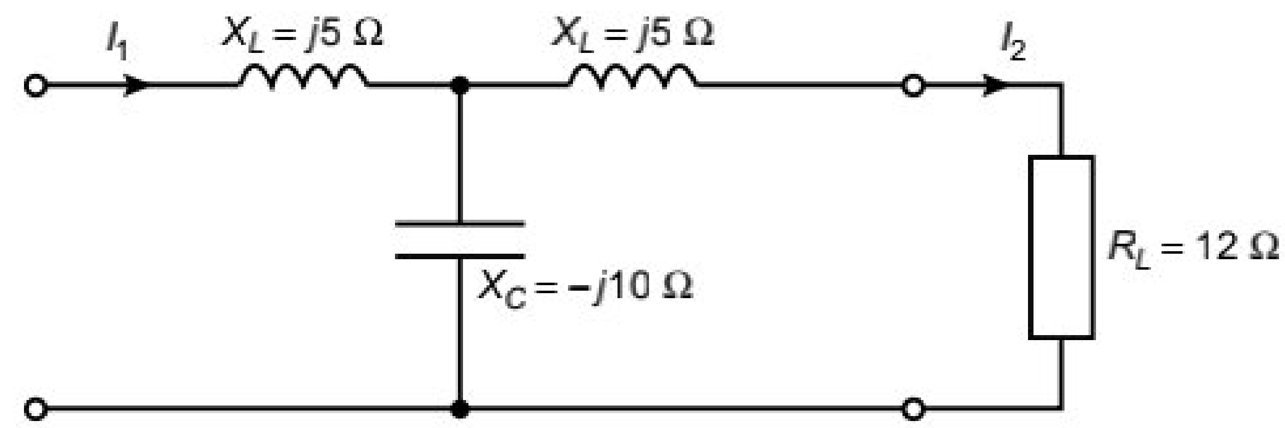


Figure 42.34

By current division in Figure 42.34, $I_2 = \left(\dfrac{X_C}{X_C + X_L + R_L}\right) I_1$

from which $\dfrac{I_1}{I_2} = \dfrac{X_C + X_L + R_L}{X_C} = \dfrac{-j10 + j5 + 12}{-j10} = \dfrac{-j5 + 12}{-j10}$

$$= \frac{-j5}{-j10} + \frac{12}{-j10}$$

$$= 0.5 + \frac{j12}{-j^2 10} = 0.5 + j1.2$$

$$= 1.3\angle 67.38^\circ \text{ or } 1.3\angle 1.176$$

From equation (42.25), $\dfrac{I_1}{I_2} = e^{\alpha}\angle\beta = 1.3\angle 1.176$

(a) The attenuation coefficient, $\alpha = \ln 1.3 = \mathbf{0.262\ N}$

(b) The phase shift coefficient, $\boldsymbol{\beta} = \mathbf{1.176\ rad}$

(c) The propagation coefficient, $\gamma = \alpha + j\beta = \mathbf{(0.262 + j1.176)}$ or $\mathbf{1.205\angle 77.44^\circ}$

Variation in phase angle in the pass-band of a filter

In practise, the low and high-pass filter sections discussed in Sections 42.5 and 42.6 would possess a phase shift between the input and output voltages which varies considerably over the range of frequency comprising the pass-band.

Let the **low-pass prototype *T* section** shown in Figure 42.35 be terminated as shown in its nominal impedance R_0. The input impedance for frequencies much less than the cut-off frequency is thus also equal to R_0 and is resistive. The phasor diagram representing Figure 42.35 is shown in Figure 42.36 and is produced as follows:

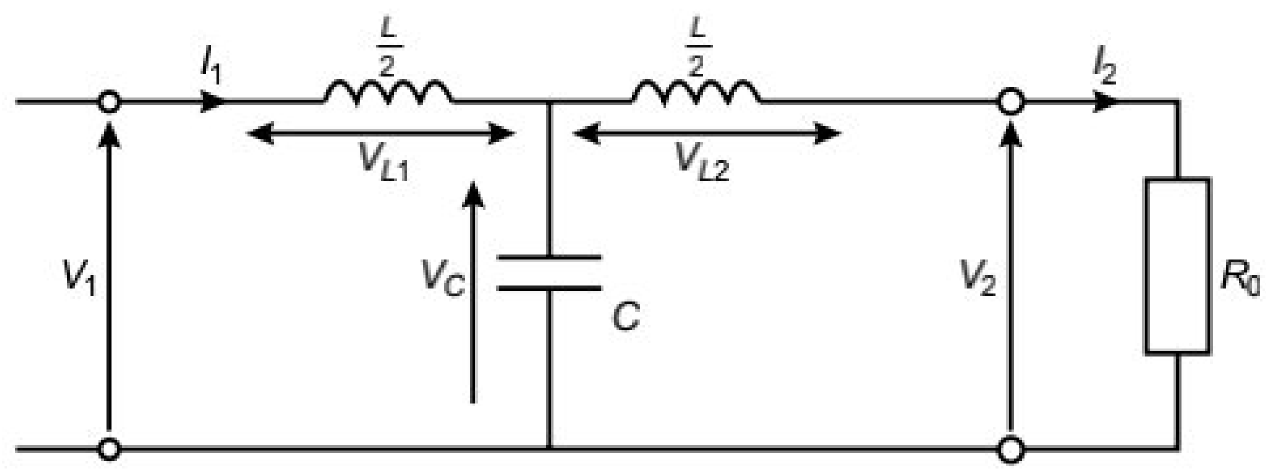


Figure 42.35

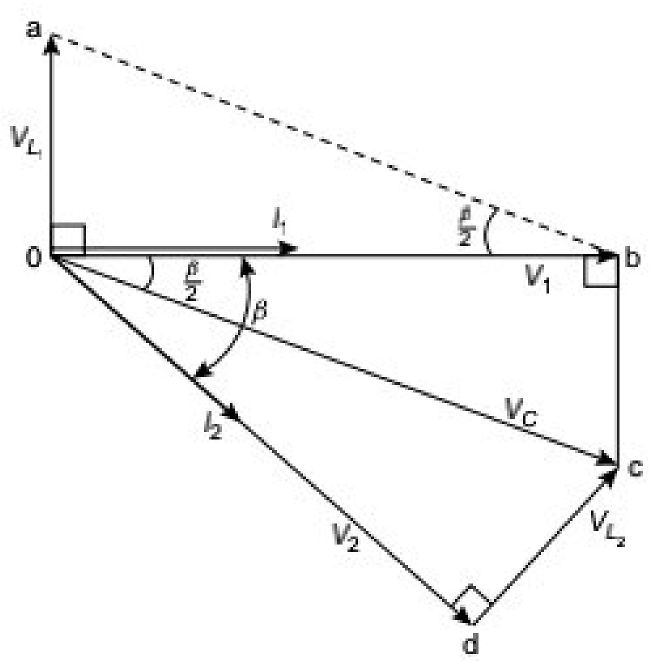


Figure 42.36

(i) V_1 and I_1 are in phase (since the input impedance is resistive).

(ii) Voltage $V_{L1} = I_1 X_L = I_1\left(\frac{\omega L}{2}\right)$, which leads I by 90°.

(iii) Voltage V_1 is the phasor sum of V_{L1} and V_C. Thus V_C is drawn as shown, completing the parallelogram oabc.

(iv) Since no power is dissipated in reactive elements $V_1 = V_2$ in magnitude.

(v) Voltage $V_{L2} = I_2\left(\frac{\omega L}{2}\right) = I_1\left(\frac{\omega L}{2}\right) = V_{L1}$

(vi) Voltage V_C is the phasor sum of V_{L2} and V_2 as shown by triangle ocd, where V_{L2} is at right angles to V_2

(vii) Current I_2 is in phase with V_2 since the output impedance is resistive. The phase lag over the section is the angle between V_1 and V_2 shown as angle β in Figure 42.36,

$$\text{where } \tan\frac{\beta}{2} = \frac{oa}{ob} = \frac{V_{L1}}{V_1} = \frac{I_1\left(\frac{\omega L}{2}\right)}{I_1 R_0} = \frac{\frac{\omega L}{2}}{R_0}$$

From equation (42.5), $R_0 = \sqrt{\frac{L}{C}}$, thus $\tan\frac{\beta}{2} = \frac{\frac{\omega L}{2}}{\sqrt{\frac{L}{C}}} = \frac{\omega\sqrt{(LC)}}{2}$

For angles of β up to about 20°, $\tan\frac{\beta}{2} \approx \frac{\beta}{2}$ radians

Thus when $\beta < 20°$, $\frac{\beta}{2} = \frac{\omega\sqrt{(LC)}}{2}$

from which, **phase angle,** $\boldsymbol{\beta = \omega\sqrt{(LC)}}$ **radian** (42.27)

Since $\beta = 2\pi f\sqrt{(LC)} = (2\pi\sqrt{(LC)})f$ then β is proportional to f and a graph of β (vertical) against frequency (horizontal) should be a straight line of gradient $2\pi\sqrt{(LC)}$ and passing through the origin. However in practise this is only usually valid up to a frequency of about 0.7 f_c for a low-pass filter and a typical characteristic is shown in Figure 42.37. At the cut-off frequency, $\beta = \pi$ rad. For frequencies within the attenuation band, the phase shift is unimportant, since all voltages having such frequencies are suppressed.

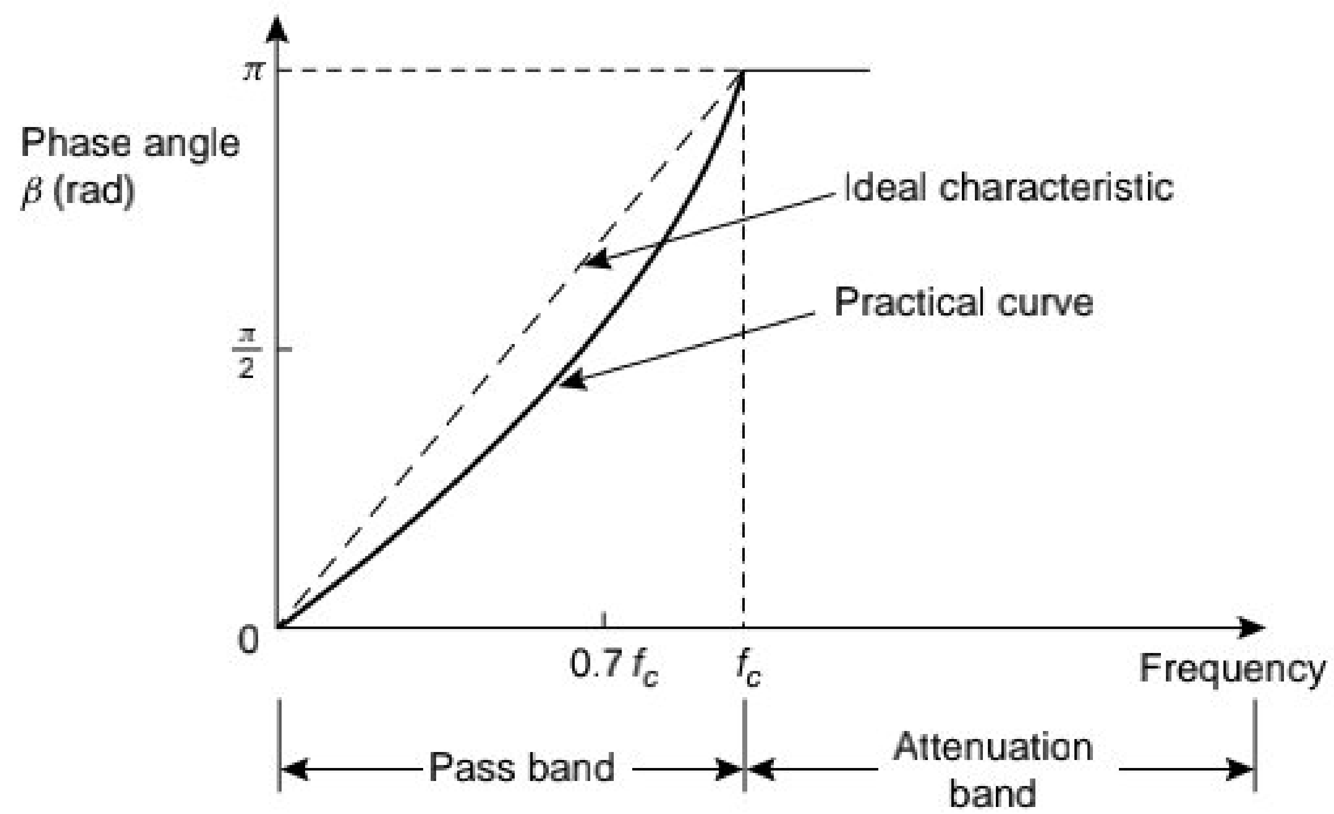


Figure 42.37

A **high-pass prototype *T* section** is shown in Figure 42.38(a) and its phasor diagram in Figure 42.38(b), the latter being produced by similar reasoning to above.

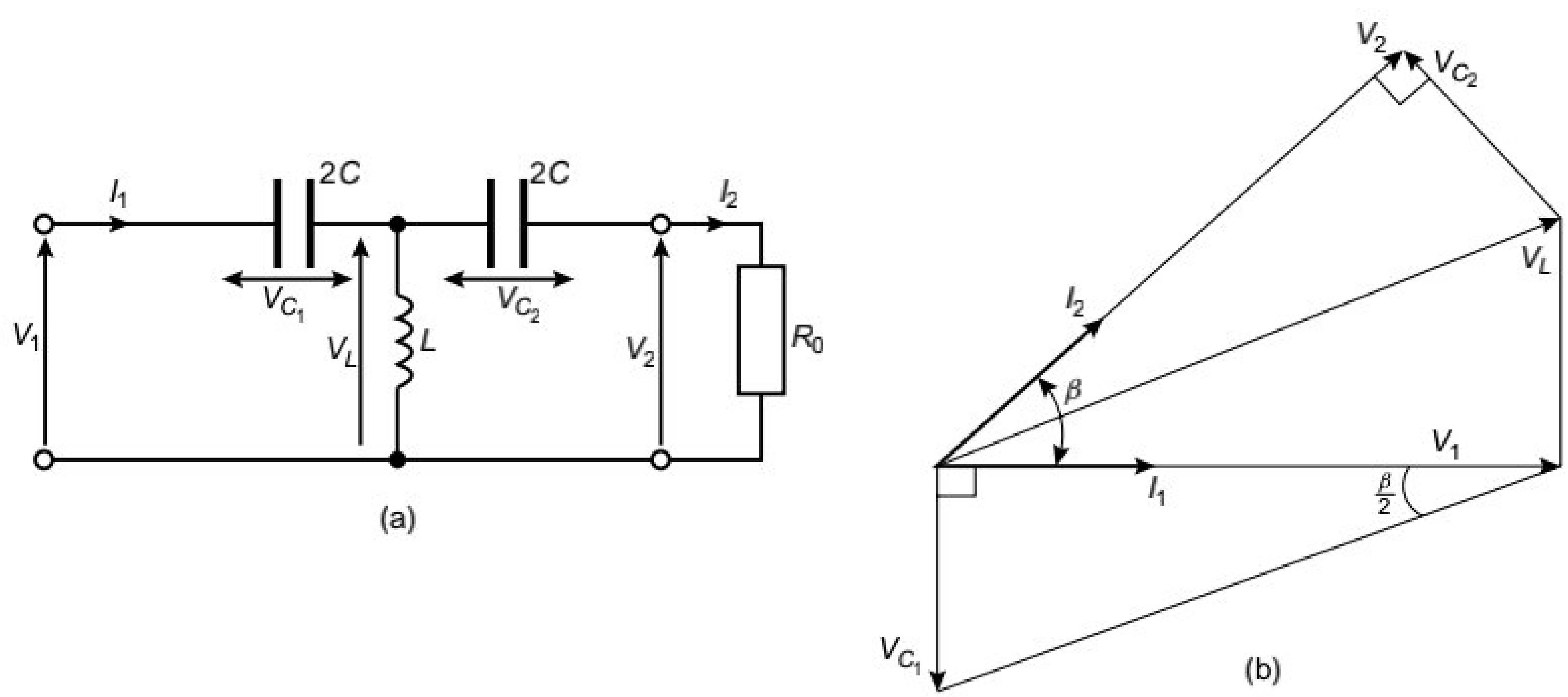


Figure 42.38

From Figure 42.38(b), $\tan\dfrac{\beta}{2} = \dfrac{V_{C1}}{V_1} = \dfrac{I_1\left(\dfrac{1}{\omega 2C}\right)}{I_1 R_0} = \dfrac{1}{2\omega C R_0}$

$$= \frac{1}{2\omega C\sqrt{\dfrac{L}{C}}} = \frac{1}{2\omega\sqrt{(LC)}}$$

i.e., $\beta = \dfrac{1}{\omega\sqrt{(LC)}} = \dfrac{1}{(2\pi\sqrt{(LC)})f}$ for small angles.

Thus the phase angle is universely proportional to frequency. The β/f characteristics of an ideal and a practical high-pass filter are shown in Figure 42.39.

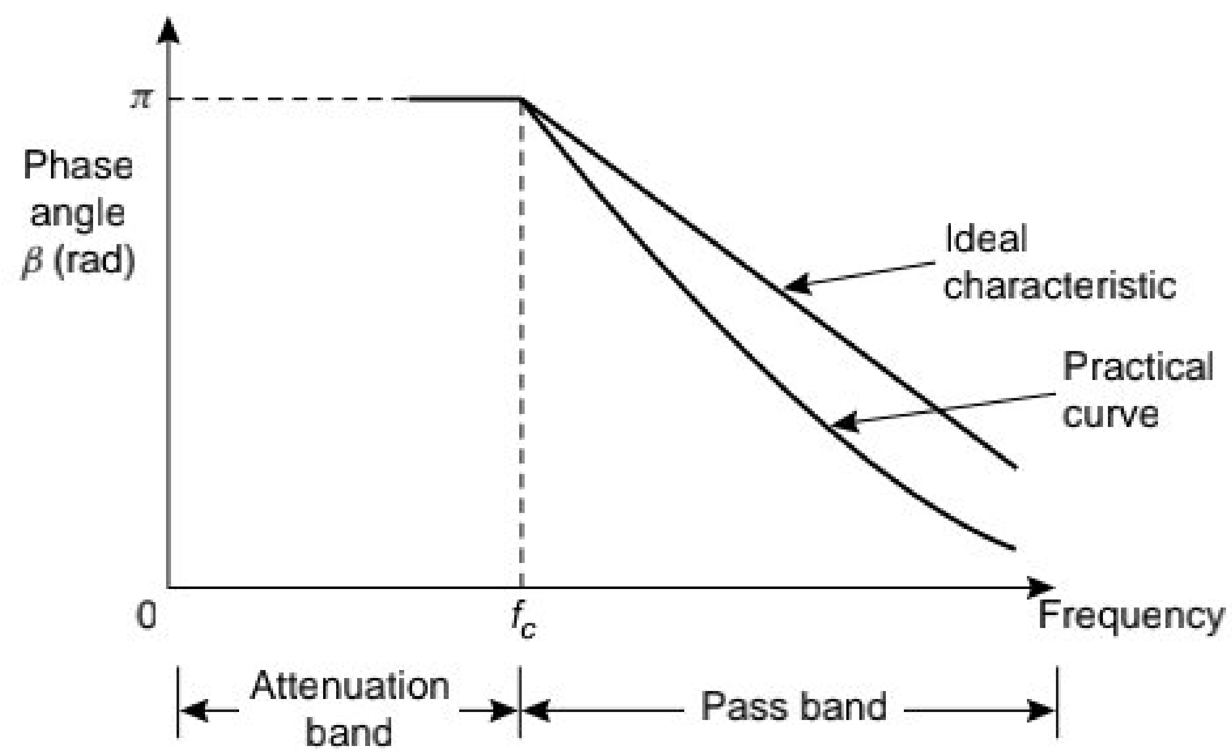


Figure 42.39

Time delay

The change of phase that occurs in a filter section depends on the time the signal takes to pass through the section. The phase shift β may be expressed as a time delay. If the frequency of the signal is f then the periodic time is $(1/f)$ seconds.

Hence the time delay $= \dfrac{\beta}{2\pi} \times \dfrac{1}{f} = \dfrac{\beta}{\omega}$.

From equation (42.27), $\beta = \omega\sqrt{(LC)}$. Thus

$$\textbf{time delay} = \frac{\omega\sqrt{(LC)}}{\omega} = \sqrt{(LC)} \qquad (42.28)$$

when angle β is small.

Equation (42.28) shows that the time delay, or **transit time**, is independant of frequency. Thus a phase shift which is proportional to frequency (equation (42.27)) results in a time delay which is independant of frequency. Hence if the input to the filter section consists of a complex wave composed of several harmonic components of differing frequency, the output will consist of a complex wave made up of the sum of corresponding components all delayed by the same amount. There will therefore be no phase distortion due to varying time delays for the separate frequency components.

In practise, however, phase shift β tends not to be constant and the increase in time delay with rising frequency causes distortion of non-sinusoidal inputs, this distortion being superimposed on that due to the attenuation of components whose frequency is higher than the cut-off frequency.

At the cut-off frequency of a prototype low-pass filter, the phase angle $\beta = \pi$ rad. Hence the time delay of a signal through such a section at the cut-off frequency is given by

$$\frac{\beta}{\omega} = \frac{\pi}{2\pi f_c} = \frac{1}{2f_c} = \frac{1}{2\dfrac{1}{\pi\sqrt{(LC)}}} \text{ from equation (42.3),}$$

i.e., at f_c, $$\boxed{\textbf{the transit time} = \frac{\pi\sqrt{(LC)}}{2} \textbf{ seconds}} \qquad (42.29)$$

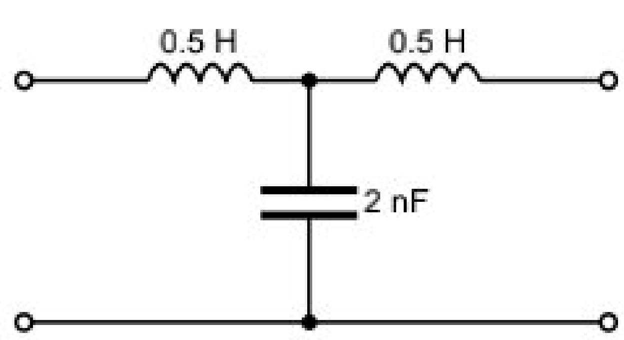


Figure 42.40

> **Problem 12.** Determine for the filter section shown in Figure 42.40, (a) the time delay for the signal to pass through the filter, assuming the phase shift is small, and (b) the time delay for a signal to pass through the section at the cut-off frequency.

Comparing Figure 42.40 with the low-pass T section of Figure 42.13(a), shows that

$$\frac{L}{2} = 0.5 \text{ H, thus inductance } L = 1 \text{ H, and capacitance } C = 2 \text{ nF}$$

(a) From equation (42.28),

$$\text{time delay} = \sqrt{(LC)} = \sqrt{[(1)(2 \times 10^{-9})]} = \mathbf{44.7\ \mu s}$$

(b) From equation (42.29), at the cut-off frequency,

$$\text{time delay} = \frac{\pi}{2}\sqrt{(LC)} = \frac{\pi}{2}(44.7) = \mathbf{70.2\ \mu s}$$

Problem 13. A filter network comprising n identical sections passes signals of all frequencies up to 500 kHz and provides a total delay of 9.55 μs. If the nominal impedance of the circuit into which the filter is inserted is 1 kΩ, determine (a) the values of the elements in each section, and (b) the value of n.

Cut-off frequency, $f_c = 500 \times 10^3$ Hz and nominal impedance

$$R_0 = 1000\ \Omega.$$

Since the filter passes frequencies up to 500 kHz then it is a low-pass filter.

(a) From equations (42.6) and (42.7), for a low-pass filter section,

$$\text{capacitance, } C = \frac{1}{\pi R_0 f_c} = \frac{1}{\pi(1000)(500 \times 10^3)} \equiv \mathbf{636.6\ pF}$$

$$\text{and inductance, } L = \frac{R_0}{\pi f_c} = \frac{1000}{\pi(500 \times 10^3)} \equiv \mathbf{636.6\ \mu H}$$

Thus if the section is a **low-pass *T* section** then the inductance in each series arm will be $(L/2) =$ **318.3 μH** and the capacitance in the shunt arm will be **636.6 pF**.

If the section is a **low-pass π section** then the inductance in the series arm will be **636.6 μH** and the capacitance in each shunt arm will be $(C/2) =$ **318.3 pF**

(b) From equation (42.28), the time delay for a single section

$$= \sqrt{(LC)} = \sqrt{[(636.6 \times 10^{-6})(636.6 \times 10^{-12})]} = 0.6366\ \mu s$$

For a time delay of 9.55 μs therefore, the number of cascaded sections required is given by

$$\frac{9.55}{0.6366} = 15, \text{ i.e., } \boldsymbol{n = 15}$$

Problem 14. A filter network consists of 8 sections in cascade having a nominal impedance of 1 kΩ. If the total delay time is 4 μs, determine the component values for each section if the filter is (a) a low-pass T network, and (b) a high-pass π network.

Since the total delay time is 4 μs then the delay time of each of the 8 sections is $\frac{4}{8}$, i.e., 0.5 μs

From equation (42.28), time delay $= \sqrt{(LC)}$

$$\text{Hence } 0.5 \times 10^{-6} = \sqrt{(LC)} \qquad \text{(i)}$$

Also, from equation (42.5), $\sqrt{\frac{L}{C}} = 1000$ (ii)

From equation (ii), $\sqrt{L} = 1000\sqrt{C}$

Substituting in equation (i) gives: $0.5 \times 10^{-6} = (1000\sqrt{C})\sqrt{C} = 1000\,C$

from which, capacitance $C = \frac{0.5 \times 10^{-6}}{1000} = 0.5$ nF

From equation (ii), $\sqrt{C} = \frac{\sqrt{L}}{1000}$

Substituting in equation (i) gives: $0.5 \times 10^{-6} = (\sqrt{L})\left(\frac{\sqrt{L}}{1000}\right) = \frac{L}{1000}$

from which, inductance, $L = 500$ μH

(a) If the filter is a **low-pass *T* section** then, from Figure 42.13(a), each series arm has an inductance of $L/2$, i.e., **250 μH** and the shunt arm has a capacitance of **0.5 nF**

(b) If the filter is a **high-pass π network** then, from Figure 42.16(b), the series arm has a capacitance of **0.5 nF** and each shunt arm has an inductance of 2 L, i.e., **1000 μH or 1 mH**.

Further problems on propagation coefficient and time delay may be found in Section 42.10, problems 13 to 18, page 838

42.8 '*m*-derived' filter sections

(a) General

In a low-pass filter a clearly defined cut-off frequency followed by a high attenuation is needed; in a high-pass filter, high attenuation followed by a clearly defined cut-off frequency is needed. It is not practicable to obtain either of these conditions by wiring appropriate prototype constant-k sections in cascade. An equivalent section is therefore required having:

(i) the same cut-off frequency as the prototype but with a rapid rise in attenuation beyond cut-off for a low-pass type or a rapid decrease at cut-off from a high attenuation for the high-pass type,

(ii) the same value of nominal impedance R_0 as the prototype at all frequencies (otherwise the two forms could not be connected together without mismatch).

If the two sections, i.e., the prototype and the equivalent section, have the same value of R_0 they will have identical pass-bands.

The equivalent section is called an **'*m*-derived' filter section** (for reasons as explained below) and is one which gives a sharper cut-off at the edges of the pass band and a better impedance characteristic.

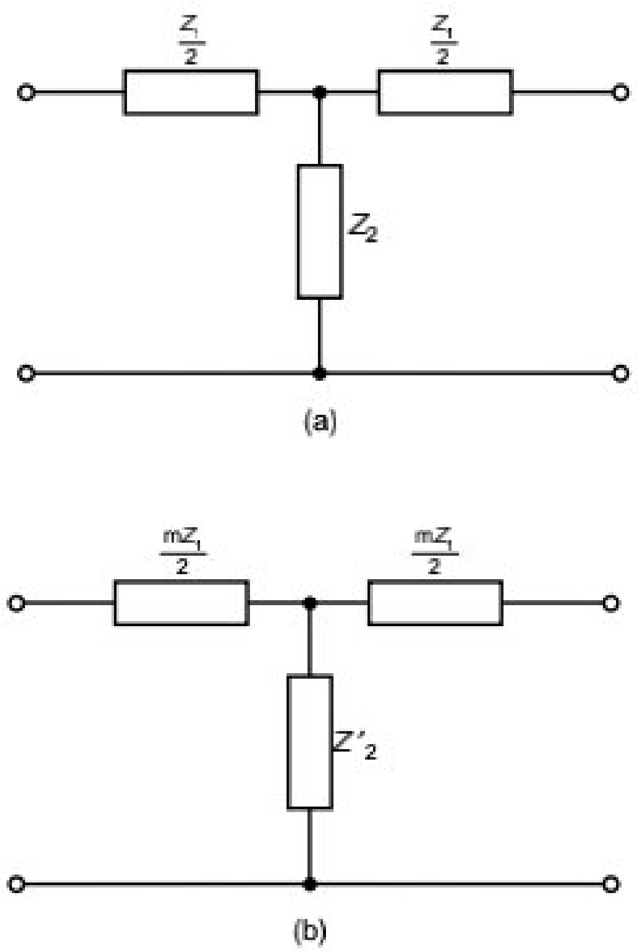


Figure 42.41

(b) *T* sections

A prototype T section is shown in Figure 42.41(a). Let a new section be constructed from this section having a series arm of the same type but of different value, say mZ_1, where m is some constant. (It is for this reason that the new equivalent section is called an 'm-derived' section.) If the characteristic impedance Z_{0T} of the two sections is to be the same then the value of the shunt arm impedance will have to be different to Z_2.

Let this be Z'_2 as shown in Figure 42.41(b).

The value of Z'_2 is determined as follows:

From equation (41.1), page 760, for the prototype shown in Figure 42.41(a):

$$Z_{0T} = \sqrt{\left[\left(\frac{Z_1}{2}\right)^2 + 2\left(\frac{Z_1}{2}\right)Z_2\right]}$$

i.e., $$Z_{0T} = \sqrt{\left(\frac{Z_1^2}{4} + Z_1Z_2\right)} \qquad (a)$$

Similarly, for the new section shown in Figure 42.41(b),

$$Z_{0T} = \sqrt{\left[\left(\frac{mZ_1}{2}\right)^2 + 2\left(\frac{mZ_1}{2}\right)Z'_2\right]}$$

i.e., $$Z_{0T} = \sqrt{\left(\frac{m^2Z_1^2}{4} + mZ_1Z'_2\right)} \qquad (b)$$

Equations (a) and (b) will be identical if:

$$\frac{Z_1^2}{4} + Z_1Z_2 = \frac{m^2Z_1^2}{4} + mZ_1Z'_2$$

Rearranging gives: $$mZ_1Z'_2 = Z_1Z_2 + \frac{Z_1^2}{4}(1 - m^2)$$

i.e., $$Z'_2 = \frac{Z_2}{m} + Z_1\left(\frac{1 - m^2}{4m}\right) \qquad (42.30)$$

Thus impedance Z'_2 consists of an impedance Z_2/m in series with an impedance $Z_1((1 - m^2)/4m)$. An additional component has therefore been introduced into the shunt arm of the m-derived section. The value of m can range from 0 to 1, and when $m = 1$, the prototype and the m-derived sections are identical.

(c) π sections

A prototype π section is shown in Figure 42.42(a). Let a new section be constructed having shunt arms of the same type but of different values,

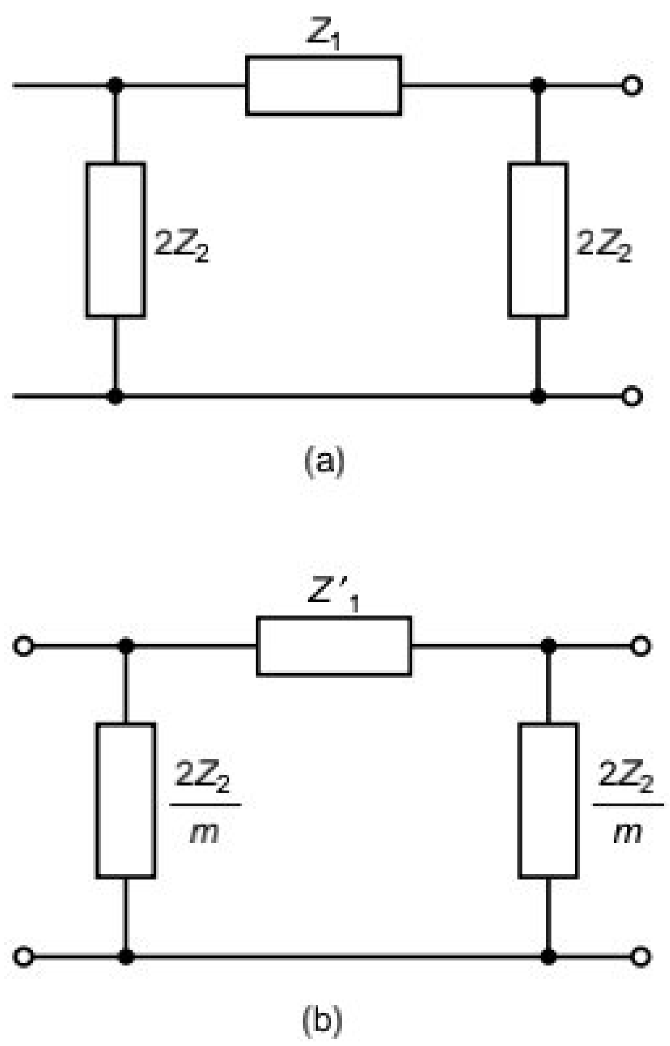


Figure 42.42

say Z_2/m, where m is some constant. If the characteristic impedance $Z_{0\pi}$ of the two sections is to be the same then the value of the series arm impedance will have to be different to Z_1.

Let this be Z'_1 as shown in Figure 42.42(b).

The value of Z'_1 is determined as follows:

From equation (42.9), $Z_{0T}Z_{0\pi} = Z_1Z_2$

Thus the characteristic impedance of the section shown in Figure 42.42(a) is given by:

$$Z_{0\pi} = \frac{Z_1Z_2}{Z_{0T}} = \frac{Z_1Z_2}{\sqrt{\left(\frac{Z_1^2}{4} + Z_1Z_2\right)}} \qquad \text{(c)}$$

from equation (a) above.

For the section shown in Figure 42.42(b),

$$Z_{0\pi} = \frac{Z'_1\frac{Z_2}{m}}{\sqrt{\left(\frac{(Z'_1)^2}{4} + Z'_1\frac{Z_2}{m}\right)}} \qquad \text{(d)}$$

Equations (c) and (d) will be identical if

$$\frac{Z_1Z_2}{\sqrt{\left(\frac{Z_1^2}{4} + Z_1Z_2\right)}} = \frac{Z'_1\frac{Z_2}{m}}{\sqrt{\left(\frac{(Z'_1)^2}{4} + Z'_1\frac{Z_2}{m}\right)}}$$

Dividing both sides by Z_2 and then squaring both sides gives:

$$\frac{Z_1^2}{\frac{Z_1^2}{4} + Z_1Z_2} = \frac{\frac{(Z'_1)^2}{m^2}}{\frac{(Z'_1)^2}{4} + \frac{Z'_1Z_2}{m}}$$

Thus $$Z_1^2\left(\frac{(Z'_1)^2}{4} + \frac{Z'_1Z_2}{m}\right) = \frac{(Z'_1)^2}{m^2}\left(\frac{Z_1^2}{4} + Z_1Z_2\right)$$

i.e., $$\frac{Z_1^2(Z'_1)^2}{4} + \frac{Z_1^2Z'_1Z_2}{m} = \frac{(Z'_1)^2Z_1^2}{4m^2} + \frac{(Z'_1)^2Z_1Z_2}{m^2}$$

Multiplying throughout by $4m^2$ gives:

$$m^2Z_1^2(Z'_1)^2 + 4mZ_1^2Z'_1Z_2 = (Z'_1)^2Z_1^2 + 4(Z'_1)^2Z_1Z_2$$

Dividing throughout by Z'_1 and rearranging gives:

$$4mZ_1^2Z_2 = Z'_1(Z_1^2 + 4Z_1Z_2 - m^2Z_1^2)$$

Thus $Z_1' = \dfrac{4mZ_1^2 Z_2}{4Z_1Z_2 + Z_1^2(1-m^2)}$

i.e., $$\mathbf{Z_1' = \frac{4mZ_1Z_2}{4Z_2 + Z_1(1-m^2)}} \qquad (42.31)$$

An impedance mZ_1 in parallel with an impedance $(4mZ_2/1-m^2)$ gives (using (product/sum)):

$$\frac{(mZ_1)\dfrac{4mZ_2}{1-m^2}}{mZ_1 + \dfrac{4mZ_2}{1-m^2}} = \frac{(mZ_1)4mZ_2}{mZ_1(1-m^2)+4mZ_2} = \frac{4mZ_1Z_2}{4Z_2 + Z_1(1-m^2)}$$

Hence the expression for Z_1' (equation (42.31)) represents an impedance mZ_1 in parallel with an impedance $(4m/1-m^2)Z_2$

(d) Low-pass '*m*-derived' sections

The '*m*-derived' low-pass T section is shown in Figure 42.43(a) and is derived from Figure 42.13(a), Figure 42.41 and equation (42.30). If Z_2 represents a pure capacitor in Figure 42.41(a), then $Z_2 = (1/\omega C)$.
A capacitance of value mC shown in Figure 42.43(a) has an impedance

$$\frac{1}{\omega mC} = \frac{1}{m}\left(\frac{1}{\omega C}\right) = \frac{Z_2}{m} \text{ as in equation (42.30).}$$

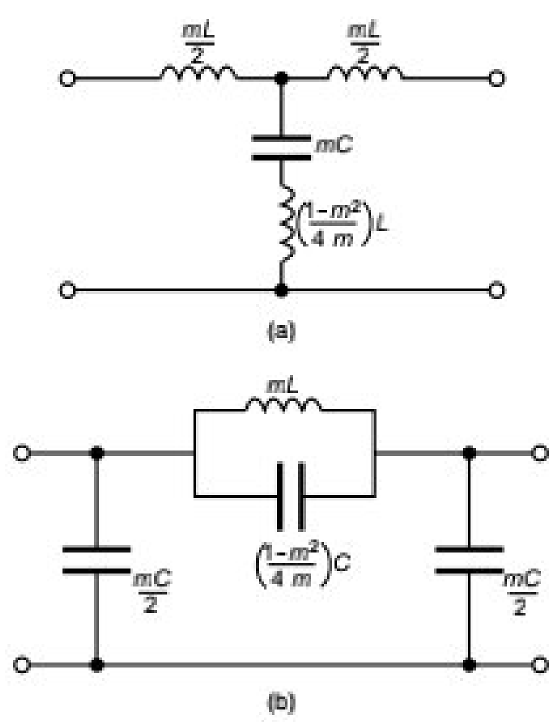


Figure 42.43

The '*m*-derived' low-pass π section is shown in Figure 42.43(b) and is derived from Figure 42.13(b), Figure 42.42 and from equation (42.31). Note that a capacitance of value $\left(\dfrac{1-m^2}{4m}\right)C$ has an impedance of

$$\frac{1}{\omega\left(\dfrac{1-m^2}{4m}\right)C} = \left(\frac{4m}{1-m^2}\right)\left(\frac{1}{\omega C}\right) = \left(\frac{4m}{1-m^2}\right)Z_2$$

where Z_2 is a pure capacitor.

In Figure 42.43(a), series resonance will occur in the shunt arm at a particular frequency — thus short-circuiting the transmission path. In the prototype, infinite attenuation is obtained only at infinite frequency (see Figure 42.25).

In the m-derived section of Figure 42.43(a), let the frequency of infinite attenuation be f_∞, then at resonance: $X_L = X_C$

i.e., $$\omega_\infty\left(\frac{1-m^2}{4m}\right)L = \frac{1}{\omega_\infty mC}$$

from which, $$\omega_\infty^2 = \frac{1}{(mC)\left(\dfrac{1-m^2}{4m}\right)L} = \frac{4}{LC(1-m^2)}$$

From equation (42.2),

$$\frac{4}{LC} = \omega_c^2, \text{ thus } \omega_\infty^2 = \frac{\omega_c^2}{(1 - m^2)},$$

where $\omega_c = 2\pi f_c$, f_c being the cut-off frequency of the prototype.

Hence $$\omega_\infty = \frac{\omega_c}{\sqrt{(1 - m^2)}} \qquad (42.32)$$

Rearranging gives:

$$\omega_\infty^2(1 - m^2) = \omega_c^2$$
$$\omega_\infty^2 - m^2\omega_\infty^2 = \omega_c^2$$
$$m^2 = \frac{\omega_\infty^2 - \omega_c^2}{\omega_\infty^2} = 1 - \frac{\omega_c^2}{\omega_\infty^2}$$

i.e., $$\boldsymbol{m = \sqrt{\left[1 - \left(\frac{f_c}{f_\infty}\right)^2\right]}} \qquad (42.33)$$

In the m-derived π section of Figure 42.43(b), resonance occurs in the parallel arrangement comprising the series arm of the section when

$$\omega^2 = \frac{1}{mL\left(\dfrac{1 - m^2}{4m}\right)C}, \text{ when } \omega^2 = \frac{4}{LC(1 - m^2)}$$

as in the series resonance case (see Chapter 28).

Thus equations (42.32) and (42.33) are also applicable to the low-pass m-derived π section.

In equation (42.33), $\mathbf{0 < m < 1}$, thus $\boldsymbol{f_\infty > f_c}$.

The frequency of infinite attenuation f_∞ can be placed anywhere within the attenuation band by suitable choice of the value of m; the smaller m is made the nearer is f_∞ to the cut-off frequency, f_c.

Problem 15. A filter section is required to have a nominal impedance of 600 Ω, a cut-off frequency of 5 kHz and a frequency of infinite attenuation at 5.50 kHz. Design (a) an appropriate 'm-derived' T section, and (b) an appropriate 'm-derived' π section.

Nominal impedance $R_0 = 600\ \Omega$, cut-off frequency, $f_c = 5000$ Hz and frequency of infinite attenuation, $f_\infty = 5500$ Hz. Since $f_\infty > f_c$ the filter section is low-pass.

From equation (42.33),

$$m = \sqrt{\left[1 - \left(\frac{f_c}{f_\infty}\right)^2\right]} = \sqrt{\left[1 - \left(\frac{5000}{5500}\right)^2\right]} = 0.4166$$

For a low-pass prototype section:

$$\text{from equation (42.6), capacitance, } C = \frac{1}{\pi R_0 f_c} = \frac{1}{\pi(600)(5000)} \equiv 0.106\ \mu\text{F}$$

$$\text{and from equation (42.7), inductance, } L = \frac{R_0}{\pi f_c} = \frac{600}{\pi(5000)} \equiv 38.2\ \text{mH}$$

(a) For an 'm-derived' low-pass T section:

From Figure 42.43(a), the series arm inductances are each

$$\frac{mL}{2} = \frac{(0.4166)(38.2)}{2} = \mathbf{7.957\ mH,}$$

and the shunt arm contains a capacitor of value mC, i.e., (0.4166)(0.106) = **0.0442 μF** or **44.2 nF**, in series with an inductance of

$$\text{value } \left(\frac{1 - m^2}{4m}\right) L = \left(\frac{1 - 0.4166^2}{4(0.4166)}\right)(38.2),$$

i.e., **18.95 mH**

The appropriate 'm-derived' T section is shown in Figure 42.44.

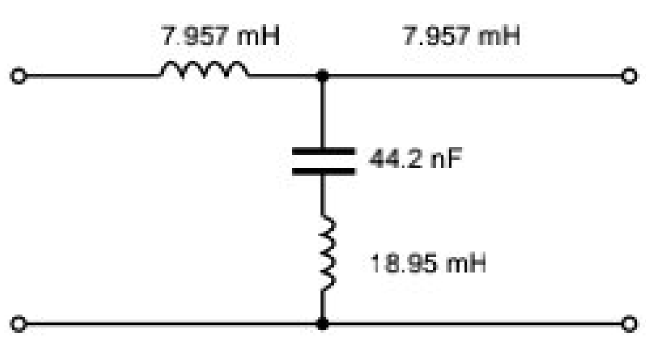


Figure 42.44

(b) For an 'm-derived' low-pass π section:

From Figure 42.43(b) the shunt arms each contain capacitances equal to $mC/2$,

$$\text{i.e., } \frac{(0.4166)(0.106)}{2} = \mathbf{0.0221\ \mu F\ or\ 22.1\ nF},$$

and the series arm contains an inductance of value $m\,L$, i.e., (0.4166)(38.2) = **15.91 mH** in parallel with a capacitance of

$$\text{value } \left(\frac{1 - m^2}{4m}\right) C = \left(\frac{1 - 0.4166^2}{4(0.4166)}\right)(0.106)$$

$$= \mathbf{0.0526\ \mu F\ or\ 52.6\ nF}$$

The appropriate 'm-derived'π section is shown in Figure 42.45.

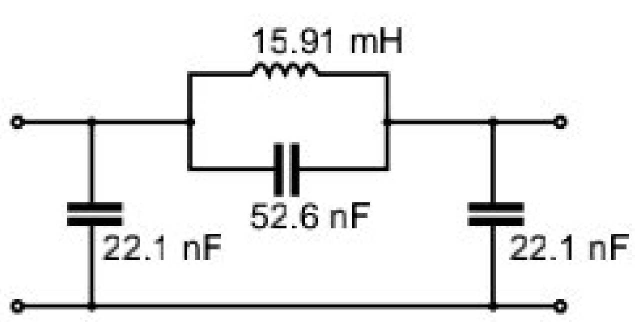


Figure 42.45

(e) High-pass '*m*-derived' sections

Figure 42.46(a) shows a high-pass prototype T section and Figure 42.46(b) shows the 'm-derived' high-pass T section which is derived from Figure 42.16(a), Figure 42.41 and equation (42.30).

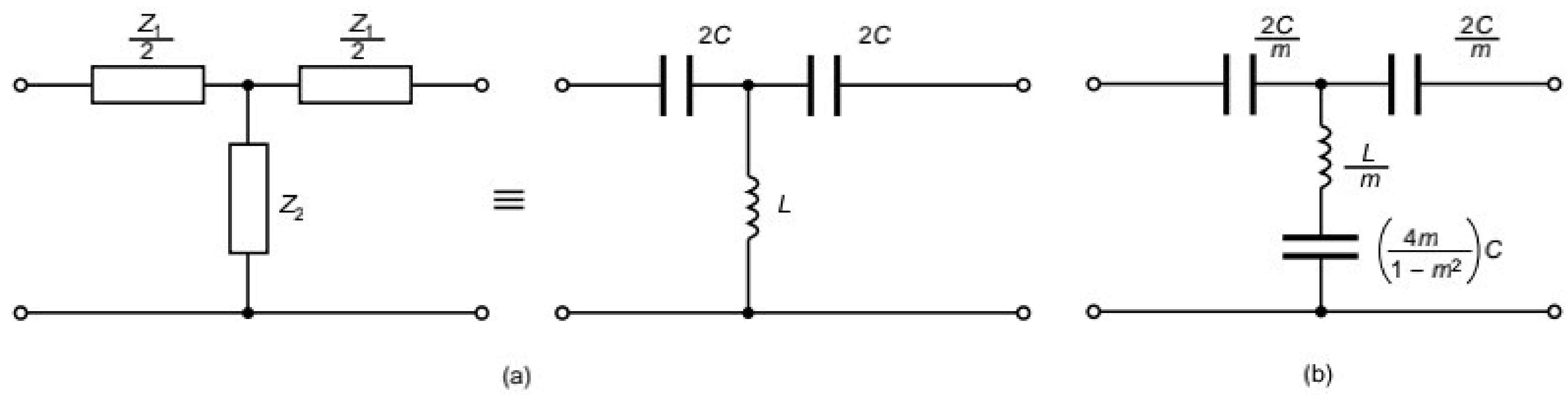


Figure 42.46

Figure 42.47(a) shows a high-pass prototype π section and Figure 42.47(b) shows the 'm-derived' high-pass π section which is derived from Figure 42.16(b), Figure 42.42 and equation (42.31). In Figure 42.46(b), resonance occurs in the shunt arm when:

$$\omega_\infty \frac{L}{m} = \frac{1}{\omega_\infty \left(\frac{4m}{1-m^2}\right) C}$$

i.e., when $\omega_\infty^2 = \dfrac{1-m^2}{4LC} = \omega_c^2(1-m^2)$ from equation (42.14)

i.e., $$\omega_\infty = \omega_c\sqrt{(1-m^2)} \quad (42.34)$$

Hence $$\frac{\omega_\infty^2}{\omega_c^2} = 1 - m^2$$

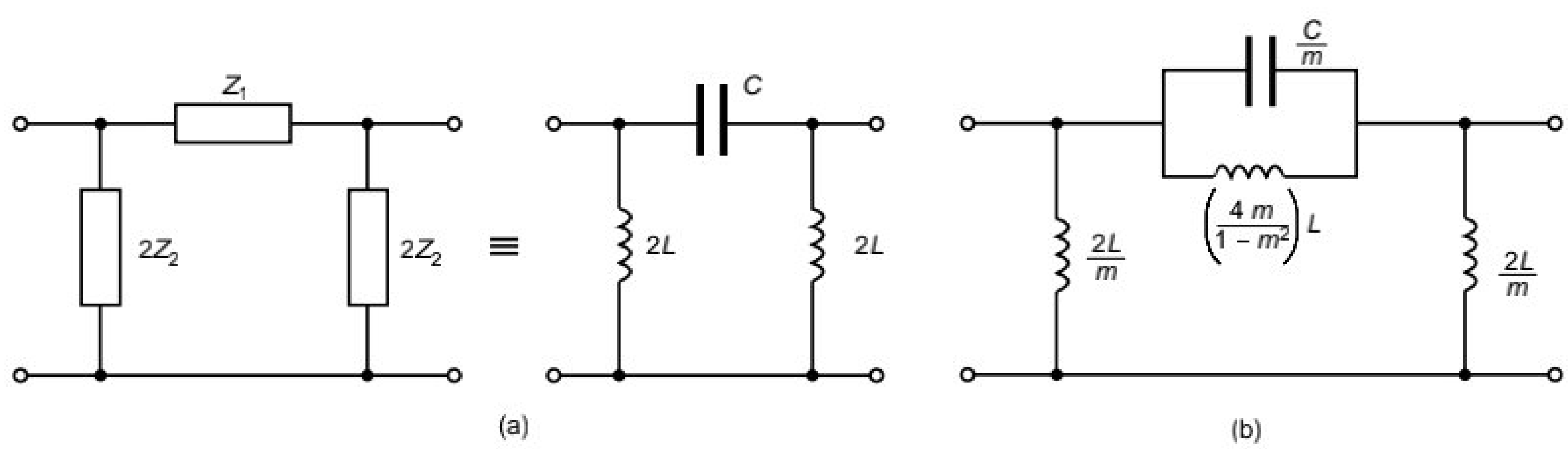


Figure 42.47

from which, $$\boldsymbol{m = \sqrt{\left[1 - \left(\frac{f_\infty}{f_c}\right)^2\right]}} \qquad (42.35)$$

For a high-pass section, $f_\infty < f_c$.

It may be shown that equations (42.34) and (42.35) also apply to the 'm-derived' π section shown in Figure 42.47(b).

Problem 16. Design (a) a suitable 'm-derived' T section, and (b) a suitable 'm-derived' π section having a cut-off frequency of 20 kHz, a nominal impedance of 500 Ω and a frequency of infinite attenuation 16 kHz.

Nominal impedance $R_0 = 500\ \Omega$, cut-off frequency, $f_c = 20$ kHz and the frequency of infinite attenuation, $f_\infty = 16$ kHz. Since $f_\infty < f_c$ the filter is high-pass.

From equation (42.35), $$m = \sqrt{\left[1 - \left(\frac{f_\infty}{f_c}\right)^2\right]} = \sqrt{\left[1 - \left(\frac{16}{20}\right)^2\right]} = 0.60$$

For a high-pass prototype section:

From equation (42.18), capacitance,

$$C = \frac{1}{4\pi R_0 f_c} = \frac{1}{4\pi(500)(20\,000)} \equiv 7.958 \text{ nF}$$

and from equation (42.19), inductance,

$$L = \frac{R_0}{4\pi f_c} = \frac{500}{4\pi(20\,000)} \equiv 1.989 \text{ mH}$$

(a) For an 'm-derived' high-pass T section:

From Figure 42.46(b), each series arm contains a capacitance of value $2C/m$, i.e., 2(7.958)/0.60, i.e., **26.53 nF**, and the shunt arm contains an inductance of value L/m, i.e., $(1.989/0.60) =$ **3.315 mH** in series with a capacitance of value

$$\left(\frac{4m}{1 - m^2}\right) C \text{ i.e., } \left(\frac{4(0.60)}{1 - 0.60^2}\right)(7.958) = \mathbf{29.84\ nF}$$

A suitable 'm-derived' T section is shown in Figure 42.48.

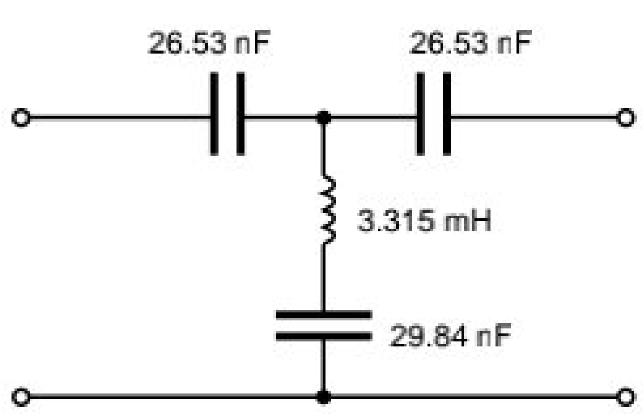


Figure 42.48

(b) For an 'm-derived' high pass π section:

From Figure 42.47(b), the shunt arms each contain inductances equal to $2L/m$, i.e., (2(1.989)/0.60), i.e., **6.63 mH** and the series arm

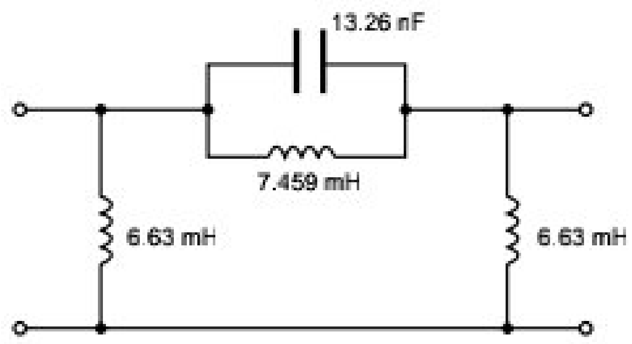


Figure 42.49

contains a capacitance of value C/m, i.e., $(7.958/0.60) = \mathbf{13.26\ nF}$ in parallel with an inductance of value $(4m/1 - m^2)L$,

i.e., $\left(\dfrac{4(0.60)}{1 - 0.60^2}\right)(1.989) \equiv \mathbf{7.459\ mH}$

A suitable 'm-derived' π section is shown in Figure 42.49.

Further problems on 'm-derived' filter sections may be found in Section 42.10, problems 19 to 22, page 839

42.9 Practical composite filters

In practise, filters to meet a given specification often have to comprise a number of basic networks. For example, a practical arrangement might consist of (i) a basic prototype, in series with (ii) an 'm-derived' section, with (iii) terminating half-sections at each end. The 'm-derived' section improves the attenuation immediately after cut-off, the prototype improves the attenuation well after cut-off, whilst the terminating half-sections are used to obtain a constant match over the pass-band.

Figure 42.50(a) shows an 'm-derived' low-pass T section, and Figure 42.50(b) shows the same section cut into two halves through AB, each of the two halves being termed a 'half-section'. The 'm-derived' half section also improves the steepness of attenuation outside the pass-band.

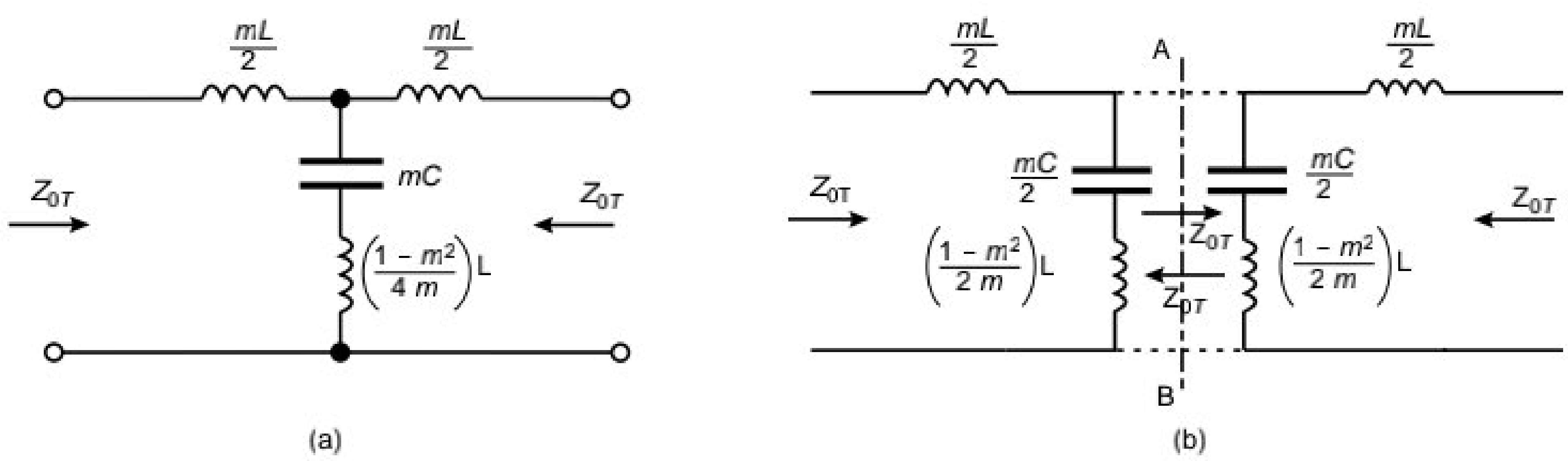


Figure 42.50

As shown in Section 42.8, the 'm-derived' filter section is based on a prototype which presents its own characteristic impedance at its terminals. Hence, for example, the prototype of a T section leads to an 'm-derived' T section and $Z_{0T} = Z_{0T(m)}$ where Z_{0T} is the characteristic impedance of the prototype and $Z_{0T(m)}$ is the characteristic impedance of the 'm-derived' section. It is shown in Figure 42.24 that Z_{0T} has a non-linear characteristic against frequency; thus $Z_{0T(m)}$ will also be non-linear.

Since from equation (42.9), $Z_{0\pi} = (Z_1 Z_2 / Z_{0T})$, then the characteristic impedance of the 'm-derived' π section,

$$Z_{0\pi(m)} = \frac{Z_1' Z_2'}{Z_{0T(m)}} = \frac{Z_1' Z_2'}{Z_{0T}}$$

where Z'_1 and Z'_2 are the equivalent values of impedance in the '*m*-derived' section.

From Figure 42.41, $Z'_1 = mZ_1$ and from equation (42.30),

$$Z'_2 = \frac{Z_2}{m} + \left(\frac{1-m^2}{4m}\right) Z_1$$

Thus $$Z_{0\pi(m)} = \frac{mZ_1\left[\dfrac{Z_2}{m} + \left(\dfrac{1-m^2}{4m}\right) Z_1\right]}{Z_{0T}}$$

$$= \frac{Z_1 Z_2}{Z_{0T}}\left[1 + \left(\frac{1-m^2}{4Z_2}\right) Z_1\right] \qquad (42.36)$$

or $$\mathbf{Z_{0\pi(m)} = Z_{0\pi}\left[1 + \left(\frac{1-m^2}{4Z_2}\right) Z_1\right]} \qquad (42.37)$$

Thus the impedance of the '*m*-derived' section is related to the impedance of the prototype by a factor of $[1 + (1 - m^2/4Z_2)Z_1]$ and will vary as m varies.

When $m = 1$, $Z_{0\pi(m)} = Z_{0\pi}$

When $m = 0$, $Z_{0\pi(m)} = \dfrac{Z_1 Z_2}{Z_{0T}}\left[1 + \dfrac{Z_1}{4Z_2}\right]$ from equation (42.36)

$$= \frac{1}{Z_{0T}}\left[Z_1 Z_2 + \frac{Z_1^2}{4}\right]$$

However from equation (42.8), $Z_1 Z_2 + \dfrac{Z_1^2}{4} = Z_{0T}^2$

Hence, when $m = 0$, $Z_{0\pi(m)} = \dfrac{Z_{0T}^2}{Z_{0T}} = Z_{0T}$

Thus the characteristic of impedance against frequency for $m = 1$ and $m = 0$ shown in Figure 42.51 are the same as shown in Figure 42.24. Further characteristics may be drawn for values of m between 0 and 1 as shown.

It is seen from Figure 42.51 that when $m = 0.6$ the impedance is practically constant at R_0 for most of the pass-band. In a composite filter, '*m*-derived' half-sections having a value of $m = 0.6$ are usually used at each end to provide a good match to a resistive source and load over the pass-band.

Figure 42.51 shows characteristics of '*m*-derived' low-pass filter sections; similar curves may be constructed for *m*-derived high-pass filters with the two curves shown in Figure 42.29 representing the limiting values of $m = 0$ and $m = 1$.

The value of m needs to be small for the frequency of input attenuation, f_∞, to be close to the cut-off frequency, f_c. However, it is not practical

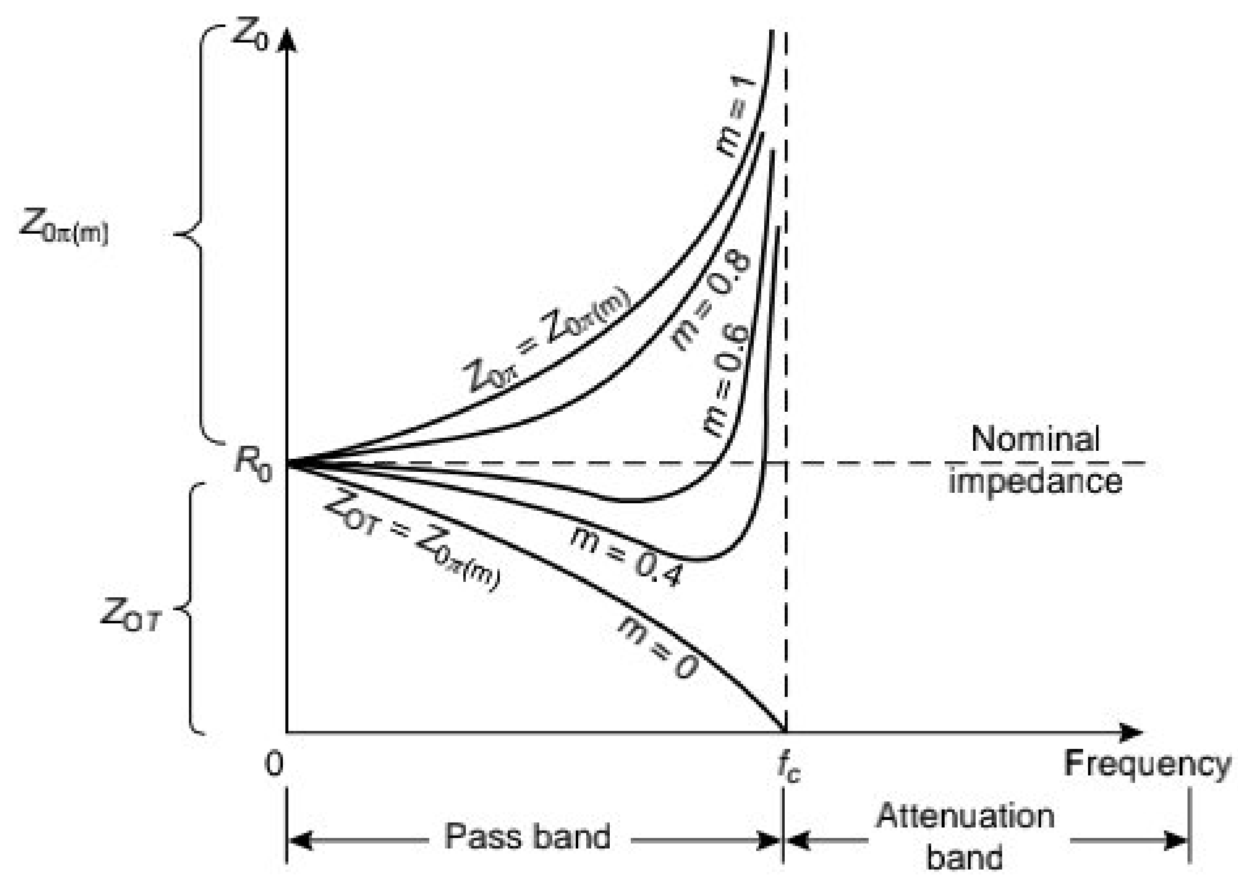


Figure 42.51

to make m very small, below 0.3 being very unusual. When $m = 0.3$, $f_\infty \approx 1.05 f_c$ (from equation (42.32)) and when $m = 0.6$, $f_\infty = 1.25 f_c$. The effect of the value of m on the frequency of infinite attenuation is shown in Figure 42.52 although the ideal curves shown would be modified a little in practise by resistance losses.

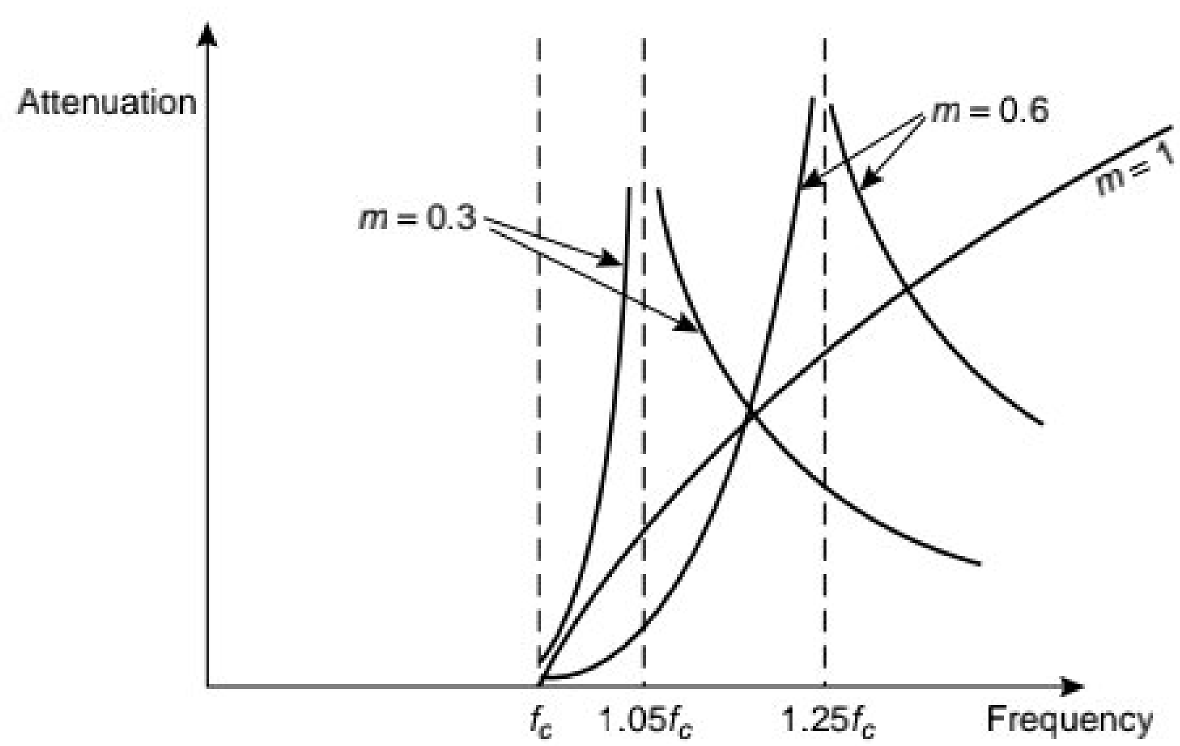


Figure 42.52

> Problem 17. It is required to design a composite filter with a cut-off frequency of 10 kHz, a frequency of infinite attenuation 11.8 kHz and nominal impedance of 600 Ω. Determine the component values needed if the filter is to comprise a prototype T section, an 'm-derived' T section and two terminating 'm-derived' half-sections.

$R_0 = 600\ \Omega$, $f_c = 10$ kHz and $f_\infty = 11.8$ kHz. Since $f_c < f_\infty$ the filter is a low-pass T section.

For the prototype:

From equation (42.6), capacitance,

$$C = \frac{1}{\pi f_c R_0} = \frac{1}{\pi(10\,000)(600)} \equiv 0.0531\ \mu\text{F},$$

and from equation (42.7), inductance,

$$L = \frac{R_0}{\pi f_c} = \frac{600}{\pi(10\,000)} \equiv 19\ \text{mH}$$

Thus, from Figure 42.13(a), the series arm components are $(L/2) = (19/2) = \mathbf{9.5\ mH}$ and the shunt arm component is $\mathbf{0.0531\ \mu F}$.

For the '*m*-derived' section:

From equation (42.33),

$$m = \sqrt{\left[1 - \left(\frac{f_c}{f_\infty}\right)^2\right]} = \sqrt{\left[1 - \left(\frac{10\,000}{11800}\right)^2\right]} = 0.5309$$

Thus from Figure 42.43(a), the series arm components are

$$\frac{mL}{2} = \frac{(0.5309)(19)}{2} = \mathbf{5.04\ mH}$$

and the shunt arm comprises $mC = (0.5309)(0.0531) = \mathbf{0.0282\ \mu F}$ in series with

$$\left(\frac{1 - m^2}{4m}\right)L = \left(\frac{1 - 0.5309^2}{4(0.5309)}\right)(19) = \mathbf{6.43\ mH}$$

For the half-sections a value of $m = 0.6$ is taken to obtain matching. Thus from Figure 42.50,

$$\frac{mL}{2} = \frac{(0.6)(19)}{2} = \mathbf{5.7\ mH}, \quad \frac{mC}{2} = \frac{(0.6)(0.0531)}{2} \equiv \mathbf{0.0159\ \mu F}$$

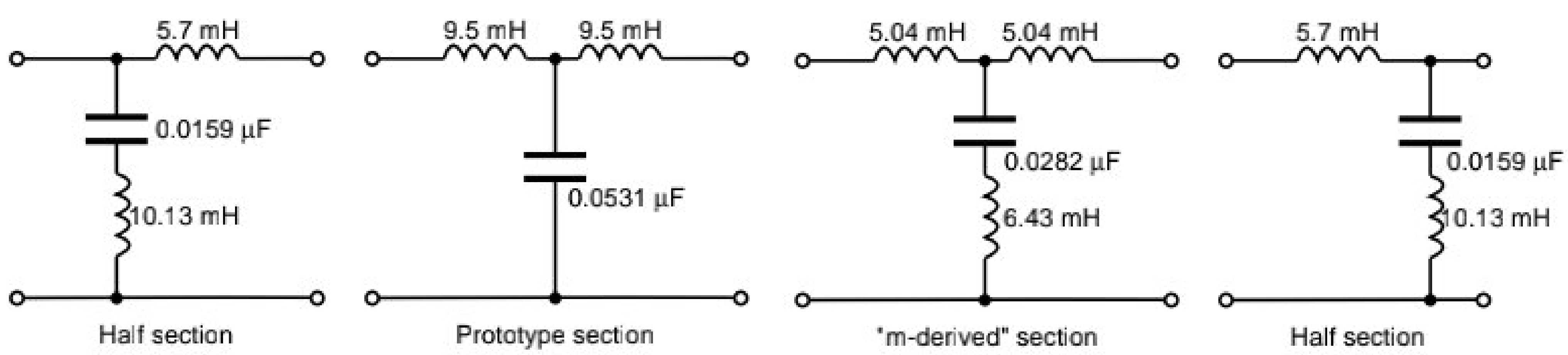


Figure 42.53

and $$\left(\frac{1-m^2}{2m}\right)L = \left(\frac{1-0.6^2}{2(0.6)}\right)(19) \equiv \mathbf{10.13\ mH}$$

The complete filter is shown in Figure 42.53.

Further problems on practical composite filter sections may be found in Section 42.10 following, problems 23 and 24, page 840

42.10 Further problems on filter networks

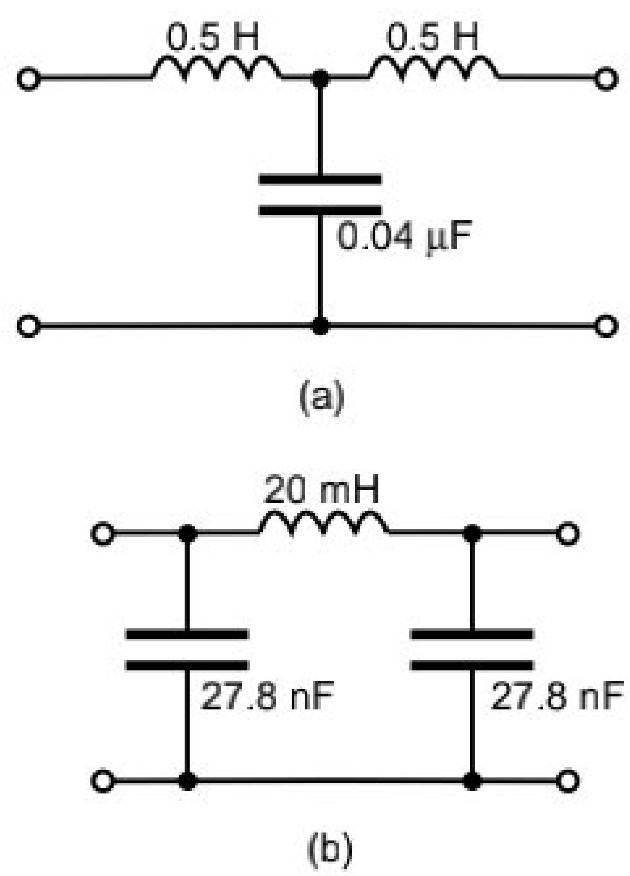


Figure 42.54

Low-pass filter sections

1 Determine the cut-off frequency and the nominal impedance of each of the low-pass filter sections shown in Figure 42.54.
[(a) 1592 Hz; 5 kΩ (b) 9545 Hz; 600 Ω]

2 A filter section is to have a characteristic impedance at zero frequency of 500 Ω and a cut-off frequency of 1 kHz. Design (a) a low-pass T section filter, and (b) a low-pass π section filter to meet these requirements.
[(a) Each series arm 79.6 mH, shunt arm 0.637 μF
(b) Series arm 159.2 mH, each shunt arm 0.318 μF]

3 Determine the value of capacitance required in the shunt arm of a low-pass T section if the inductance in each of the series arms is 40 mH and the cut-off frequency of the filter is 2.5 kHz.
[0.203 μF]

4 The nominal impedance of a low-pass π section filter is 600 Ω and its cut-off frequency is at 25 kHz. Determine (a) the value of the characteristic impedance of the section at a frequency of 20 kHz and (b) the value of the characteristic impedance of the equivalent low-pass T section filter. [(a) 1 kΩ (b) 360 Ω]

5 The nominal impedance of a low-pass π section filter is 600 Ω. If the capacitance in each of the shunt arms is 0.1 μF determine the inductance in the series arm. Make a sketch of the ideal and the practical attenuation/frequency characteristic expected for such a filter section. [72 mH]

6 A low-pass T section filter has a nominal impedance of 600 Ω and a cut-off frequency of 10 kHz. Determine the frequency at which the characteristic impedance of the section is (a) zero, (b) 300 Ω, (c) 600 Ω [(a) 10 kHz (b) 8.66 kHz (c) 0]

High-pass filter sections

7 Determine for each of the high-pass filter sections shown in Figure 42.55 (i) the cut-off frequency, and (ii) the nominal impedance.
[(a) (i) 22.51 kHz (ii) 14.14 kΩ (b) (i) 281.3 Hz (ii) 1414 Ω]

44 Transmission lines

At the end of this chapter you should be able to:

- appreciate the purpose of a transmission line
- define the transmission line primary constants R, L, C and G
- calculate phase delay, wavelength and velocity of propagation on a transmission line
- appreciate current and voltage relationships on a transmission line
- define the transmission line secondary line constants Z_0, γ, α and β
- calculate characteristic impedance and propagation coefficient in terms of the primary line constants
- understand and calculate distortion on transmission lines
- understand wave reflection and calculate reflection coefficient
- understand standing waves and calculate standing wave ratio

44.1 Introduction

A transmission line is a system of conductors connecting one point to another and along which electromagnetic energy can be sent. Thus telephone lines and power distribution lines are typical examples of transmission lines; in electronics, however, the term usually implies a line used for the transmission of radio-frequency (r.f.) energy such as that from a radio transmitter to the antenna.

An important feature of a transmission line is that it should guide energy from a source at the sending end to a load at the receiving end without loss by radiation. One form of construction often used consists of two similar conductors mounted close together at a constant separation. The two conductors form the two sides of a balanced circuit and any radiation from one of them is neutralized by that from the other. Such twin-wire lines are used for carrying high r.f. power, for example, at transmitters. The coaxial form of construction is commonly employed for low power use, one conductor being in the form of a cylinder which surrounds the other at its centre, and thus acts as a screen. Such cables are often used to couple f.m. and television receivers to their antennas.

At frequencies greater than 1000 MHz, transmission lines are usually in the form of a waveguide which may be regarded as coaxial lines without the centre conductor, the energy being launched into the guide or abstracted from it by probes or loops projecting into the guide.

44.2 Transmission line primary constants

Let an a.c. generator be connected to the input terminals of a pair of parallel conductors of infinite length. A sinusoidal wave will move along the line and a finite current will flow into the line. The variation of voltage with distance along the line will resemble the variation of applied voltage with time. The moving wave, sinusoidal in this case, is called a voltage **travelling wave**. As the wave moves along the line the capacitance of the line is charged up and the moving charges cause magnetic energy to be stored. Thus the propagation of such an **electromagnetic wave** constitutes a flow of energy.

After sufficient time the magnitude of the wave may be measured at any point along the line. The line does not therefore appear to the generator as an open circuit but presents a definite load Z_0. If the sending-end voltage is V_S and the sending end current is I_S then $Z_0 = V_S/I_S$. Thus all of the energy is absorbed by the line and the line behaves in a similar manner to the generator as would a single 'lumped' impedance of value Z_0 connected directly across the generator terminals.

There are **four parameters** associated with transmission lines, these being resistance, inductance, capacitance and conductance.

(i) **Resistance *R*** is given by $R = \rho l/A$, where ρ is the resistivity of the conductor material, A is the cross-sectional area of each conductor and l is the length of the conductor (for a two-wire system, l represents twice the length of the line). Resistance is stated in ohms per metre length of a line and represents the imperfection of the conductor. A resistance stated in ohms per loop metre is a little more specific since it takes into consideration the fact that there are two conductors in a particular length of line.

(ii) **Inductance *L*** is due to the magnetic field surrounding the conductors of a transmission line when a current flows through them. The inductance of an isolated twin line is considered in Section 40.7. From equation (40.23), page 748, the inductance L is given by

$$L = \frac{\mu_0 \mu_r}{\pi}\left\{\frac{1}{4} + \ln\frac{D}{a}\right\} \text{ henry/metre}$$

where D is the distance between centres of the conductor and a is the radius of each conductor. In most practical lines $\mu_r = 1$. An inductance stated in henrys per loop metre takes into consideration the fact that there are two conductors in a particular length of line.

(iii) **Capacitance *C*** exists as a result of the electric field between conductors of a transmission line. The capacitance of an isolated twin line is considered in Section 40.3. From equation (40.14), page 736, the capacitance between the two conductors is given by

$$C = \frac{\pi \varepsilon_0 \varepsilon_r}{\ln(D/a)} \text{ farads/metre}$$

In most practical lines $\varepsilon_r = 1$

(iv) **Conductance *G*** is due to the insulation of the line allowing some current to leak from one conductor to the other. Conductance is measured in siemens per metre length of line and represents the imperfection of the insulation. Another name for conductance is leakance.

Each of the four transmission line constants, R, L, C and G, known as the **primary constants**, are uniformly distributed along the line.

From Chapter 41, when a symmetrical T-network is terminated in its characteristic impedance Z_0, the input impedance of the network is also equal to Z_0. Similarly, if a number of identical T-sections are connected in cascade, the input impedance of the network will also be equal to Z_0.

A transmission line can be considered to consist of a network of a very large number of cascaded T-sections each a very short length (δl) of transmission line, as shown in Figure 44.1. This is an approximation of the uniformly distributed line; the larger the number of lumped parameter sections, the nearer it approaches the true distributed nature of the line. When the generator V_S is connected, a current I_S flows which divides between that flowing through the leakage conductance G, which is lost, and that which progressively charges each capacitor C and which sets up the voltage travelling wave moving along the transmission line. The loss or attenuation in the line is caused by both the conductance G and the series resistance R.

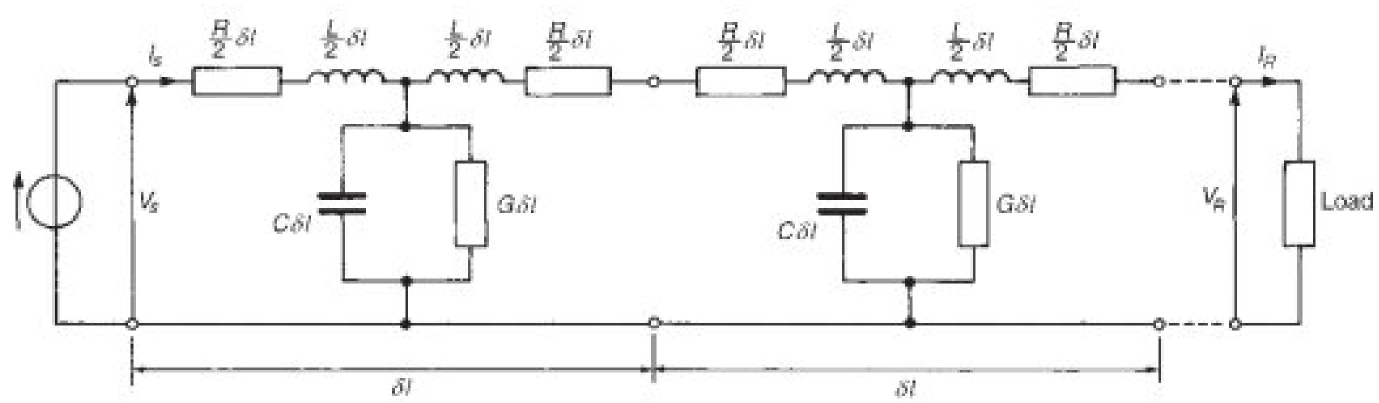


Figure 44.1

44.3 Phase delay, wavelength and velocity of propagation

Each section of that shown in Figure 44.1 is simply a low-pass filter possessing losses R and G. If losses are neglected, and R and G are removed, the circuit simplifies and the infinite line reduces to a repetitive T-section low-pass filter network as shown in Figure 44.2. Let a generator be connected to the line as shown and let the voltage be rising to a maximum positive value just at the instant when the line is connected to it. A current I_S flows through inductance L_1 into capacitor C_1. The capacitor charges and a voltage develops across it. The voltage sends a current through inductance L_1' and L_2 into capacitor C_2. The capacitor

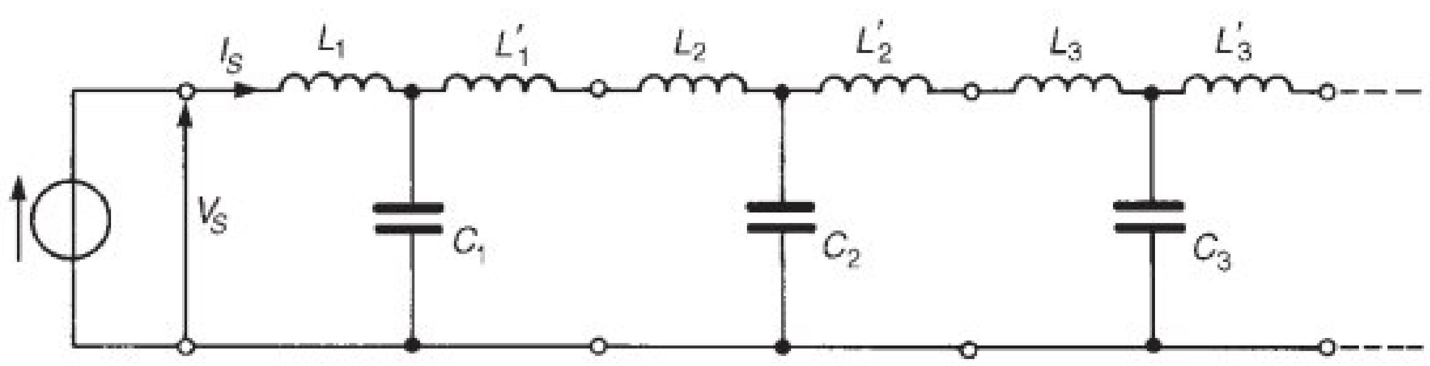


Figure 44.2

charges and the voltage developed across it sends a current through L_2' and L_3 into C_3, and so on. Thus all capacitors will in turn charge up to the maximum input voltage. When the generator voltage falls, each capacitor is charged in turn in opposite polarity, and as before the input charge is progressively passed along to the next capacitor. In this manner voltage and current waves travel along the line together and depend on each other.

The process outlined above takes time; for example, by the time capacitor C_3 has reached its maximum voltage, the generator input may be at zero or moving towards its minimum value. There will therefore be a time, and thus a phase difference between the generator input voltage and the voltage at any point on the line.

Phase delay

Since the line shown in Figure 44.2 is a ladder network of low-pass T-section filters, it is shown in equation (42.27), page 820, that the phase delay, β, is given by:

$$\boldsymbol{\beta = \omega\sqrt{(LC)}\ \textbf{radians/metre}} \qquad (44.1)$$

where L and C are the inductance and capacitance per metre of the line.

Wavelength

The wavelength λ on a line is the distance between a given point and the next point along the line at which the voltage is the same phase, the initial point leading the latter point by 2π radian. Since in one wavelength a phase change of 2π radians occurs, the phase change per metre is $2\pi/\lambda$. Hence, phase change per metre, $\beta = 2\pi/\lambda$

or $$\textbf{wavelength}, \boldsymbol{\lambda = \frac{2\pi}{\beta}}\ \textbf{metres} \qquad (44.2)$$

Velocity of propagation

The velocity of propagation, u, is given by $u = f\lambda$, where f is the frequency and λ the wavelength. Hence

$$\boldsymbol{u = f\lambda = f(2\pi/\beta) = \frac{2\pi f}{\beta} = \frac{\omega}{\beta}} \qquad (44.3)$$

The velocity of propagation of free space is the same as that of light, i.e., approximately 300×10^6 m/s. The velocity of electrical energy along a line is always less than the velocity in free space. The wavelength λ of radiation in free space is given by $\lambda = c/f$ where c is the velocity of light. Since the velocity along a line is always less than c, the wavelength corresponding to any particular frequency is always shorter on the line than it would be in free space.

> Problem 1. A parallel-wire air-spaced transmission line operating at 1910 Hz has a phase shift of 0.05 rad/km. Determine (a) the wavelength on the line, and (b) the speed of transmission of a signal.

(a) From equation (44.2), wavelength $\lambda = 2\pi/\beta = 2\pi/0.05$

$$= \mathbf{125.7\ km}$$

(b) From equation (44.3), speed of transmission,

$$u = f\lambda = (1910)(125.7) = \mathbf{240 \times 10^3\ km/s} \text{ or } \mathbf{240 \times 10^6\ m/s}$$

> Problem 2. A transmission line has an inductance of 4 mH/loop km and a capacitance of 0.004 μF/km. Determine, for a frequency of operation of 1 kHz, (a) the phase delay, (b) the wavelength on the line, and (c) the velocity of propagation (in metres per second) of the signal.

(a) From equation (44.1), phase delay,

$$\beta = \omega\sqrt{(LC)} = (2\pi 1000)\sqrt{[(4 \times 10^{-3})(0.004 \times 10^{-6})]}$$
$$= \mathbf{0.025\ rad/km}$$

(b) From equation (44.2), wavelength $\lambda = 2\pi/\beta = 2\pi/0.025$

$$= \mathbf{251\ km}$$

(c) From equation (44.3), velocity of propagation,

$$u = f\lambda = (1000)(251) \text{ km/s} = \mathbf{251 \times 10^6\ m/s}$$

Further problems on phase delay, wavelength and velocity of propagation may be found in Section 44.9, problems 1 to 3, page 897.

44.4 Current and voltage relationships

Figure 44.3 shows a voltage source V_S applied to the input terminals of an infinite line, or a line terminated in its characteristic impedance, such that a current I_S flows into the line. At a point, say, 1 km down the line let the current be I_1. The current I_1 will not have the same magnitude as I_S because of line attenuation; also I_1 will lag I_S by some angle β. The ratio I_S/I_1 is therefore a phasor quantity. Let the current a further 1 km

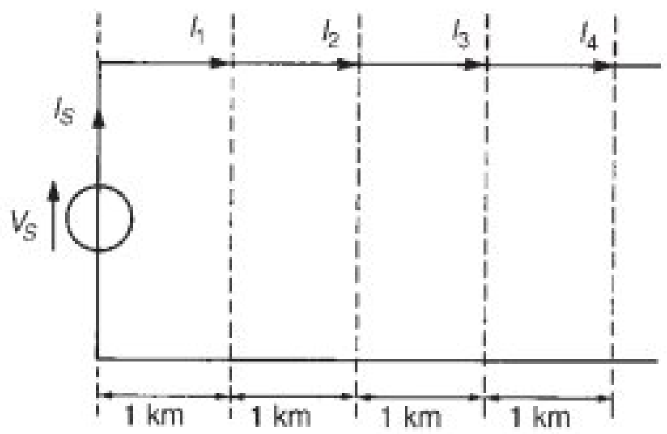


Figure 44.3

down the line be I_2, and so on, as shown in Figure 44.3. Each unit length of line can be treated as a section of a repetitive network, as explained in Section 44.2. The attenuation is in the form of a logarithmic decay and

$$\frac{I_S}{I_1} = \frac{I_1}{I_2} = \frac{I_2}{I_3} = e^{\gamma}$$

where γ is the **propagation constant**, first introduced in Section 42.7, page 815. γ has no unit.

The propagation constant is a complex quantity given by $\gamma = \alpha + j\beta$, where α is the **attenuation constant**, whose unit is the neper, and β is the **phase shift coefficient**, whose unit is the radian. For n such 1 km sections, $I_S/I_R = e^{n\gamma}$ where I_R is the current at the receiving end.

Hence $\dfrac{I_S}{I_R} = e^{n(\alpha + j\beta)} = e^{(n\alpha + jn\beta)} = e^{n\alpha}\angle n\beta$

from which, $$\boxed{\boldsymbol{I_R = I_S e^{-n\gamma} = I_S e^{-n\alpha}\angle -n\beta}} \qquad (44.4)$$

In equation (44.4), the attenuation on the line is given by $n\alpha$ nepers and the phase shift is $n\beta$ radians.

At all points along an infinite line, the ratio of voltage to current is Z_0, the characteristic impedance. Thus from equation (44.4) it follows that:

receiving end voltage, $$\boxed{\boldsymbol{V_R = V_S e^{-n\gamma} = V_S e^{-n\alpha}\angle -n\beta}} \qquad (44.5)$$

Z_0, γ, α, and β are referred to as the **secondary line constants** or **coefficients.**

> Problem 3. When operating at a frequency of 2 kHz, a cable has an attenuation of 0.25 Np/km and a phase shift of 0.20 rad/km. If a 5 V rms signal is applied at the sending end, determine the voltage at a point 10 km down the line, assuming that the termination is equal to the characteristic impedance of the line.

Let V_R be the voltage at a point n km from the sending end, then from equation (44.5), $V_R = V_S e^{-n\gamma} = V_S e^{-n\alpha}\angle -n\beta$

Since $\alpha = 0.25$ Np/km, $\beta = 0.20$ rad/km, $V_S = 5$ V and $n = 10$ km, then

$$V_R = (5)e^{-(10)(0.25)}\angle -(10)(0.20) = 5e^{-2.5}\angle -2.0 \text{ V}$$

$= \mathbf{0.41\angle -2.0\ V}$ or $\mathbf{0.41\angle -114.6^\circ\ V}$

5 V by 2.0 rad or 114.6

Thus the voltage 10 km down the line is 0.41 V rms lagging the sending end voltage o °

Problem 4. A transmission line 5 km long has a characteristic impedance of $800\angle -25^\circ\ \Omega$. At a particular frequency, the attenuation coefficient of the line is 0.5 Np/km and the phase shift coefficient is 0.25 rad/km. Determine the magnitude and phase of the current at the receiving end, if the sending end voltage is $2.0\angle 0^\circ$ V r.m.s.

The receiving end voltage (from equation (44.5)) is given by:

$$V_R = V_S e^{-n\gamma} = V_S e^{-n\alpha}\angle -n\beta = (2.0\angle 0^\circ)e^{-(5)(0.5)}\angle -(5)(0.25)$$

$$= 2.0e^{-2.5}\angle -1.25 = 0.1642\angle -71.62^\circ \text{ V}$$

Receiving end current,

$$I_R = \frac{V_R}{Z_0} = \frac{0.1642\angle -71.62^\circ}{800\angle -25^\circ} = 2.05 \times 10^{-4}\angle(-71.62^\circ - (-25^\circ))\text{A}$$

$$= \mathbf{0.205\angle -46.62^\circ\ mA}$$

Problem 5. The voltages at the input and at the output of a transmission line properly terminated in its characteristic impedance are 8.0 V and 2.0 V rms respectively. Determine the output voltage if the length of the line is doubled.

The receiving-end voltage V_R is given by $V_R = V_S e^{-n\gamma}$.

Hence $2.0 = 8.0e^{-n\gamma}$, from which, $e^{-n\gamma} = 2.0/8.0 = 0.25$

If the line is doubled in length, then

$$V_R = 8.0e^{-2n\gamma} = 8.0(e^{-n\gamma})^2$$

$$= 8.0(0.25)^2 = \mathbf{0.50\ V}$$

Further problems on current and voltage relationships may be found in Section 44.9, problems 4 to 6, page 897.

44.5 Characteristic impedance and propagation coefficient in terms of the primary constants

Characteristic impedance

At all points along an infinite line, the ratio of voltage to current is called the characteristic impedance Z_0. The value of Z_0 is independent of the length of the line; it merely describes a property of a line that is a function of the physical construction of the line. Since a short length of line may be considered as a ladder of identical low-pass filter sections, the characteristic impedance may be determined from equation (41.2), page 760, i.e.,

$$\mathbf{Z_0 = \sqrt{(Z_{OC} Z_{SC})}} \qquad (44.6)$$

since the open-circuit impedance Z_{OC} and the short-circuit impedance Z_{SC} may be easily measured.

Problem 6. At a frequency of 1.5 kHz the open-circuit impedance of a length of transmission line is $800\angle-50°\ \Omega$ and the short-circuit impedance is $413\angle-20°\ \Omega$. Determine the characteristic impedance of the line at this frequency.

From equation (44.6),

$$\text{characteristic impedance } Z_0 = \sqrt{(Z_{OC}Z_{SC})}$$
$$= \sqrt{[(800\angle-50°)(413\angle-20°)]}$$
$$= \sqrt{(330400\angle-70°)} = \mathbf{575\angle-35°\ \Omega}$$

by de Moivre's theorem.

The characteristic impedance of a transmission line may also be expressed in terms of the primary constants, R, L, G and C. Measurements of the primary constants may be obtained for a particular line and manufacturers usually state them for a standard length.

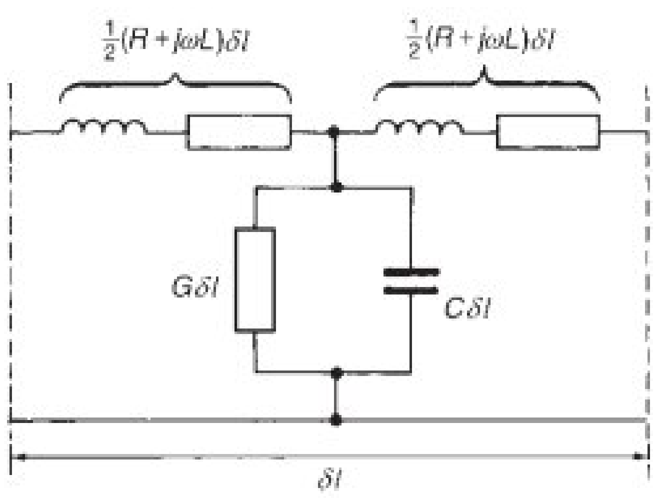


Figure 44.4

Let a very short length of line δl metres be as shown in Figure 44.4 comprising a single T-section. Each series arm impedance is $Z_1 = \frac{1}{2}(R + j\omega L)\delta l$ ohms, and the shunt arm impedance is

$$Z_2 = \frac{1}{Y_2} = \frac{1}{(G + j\omega C)\delta l}$$

[i.e., from Chapter 25, the total admittance Y_2 is the sum of the admittance of the two parallel arms, i.e., in this case, the sum of

$$G\delta l \text{ and } \left(\frac{1}{1/(j\omega C)}\right)\delta l]$$

From equation (41.1), page 760, the characteristic impedance Z_0 of a T-section having in each series arm an impedance Z_1 and a shunt arm impedance Z_2 is given by: $Z_0 = \sqrt{(Z_1{}^2 + 2Z_1Z_2)}$
Hence the characteristic impedance of the section shown in Figure 44.4 is

$$Z_0 = \sqrt{\left\{\left[\frac{1}{2}(R + j\omega L)\delta l\right]^2 + 2\left[\frac{1}{2}(R + j\omega L)\delta l\right]\left[\frac{1}{(G + j\omega C)\delta l}\right]\right\}}$$

The term $Z_1{}^2$ involves δl^2 and, since δl is a very short length of line, δl^2 is negligible. Hence

$$\boldsymbol{Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}\ \textbf{ohms}} \qquad (44.7)$$

If losses R and G are neglected, then

$$\boxed{\mathbf{Z_0 = \sqrt{(L/C)}\ ohms}} \qquad (44.8)$$

Problem 7. A transmission line has the following primary constants: resistance $R = 15\ \Omega$/loop km, inductance $L = 3.4$ mH/loop km, conductance $G = 3\ \mu$S/km and capacitance $C = 10$ nF/km. Determine the characteristic impedance of the line when the frequency is 2 kHz.

From equation (44.7),

$$\text{characteristic impedance } Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \text{ ohms}$$

$$R + j\omega L = 15 + j(2\pi 2000)(3.4 \times 10^{-3})$$
$$= (15 + j42.73)\Omega = 45.29\angle 70.66^\circ\ \Omega$$
$$G + j\omega C = 3 \times 10^{-6} + j(2\pi 2000)(10 \times 10^{-9})$$
$$= (3 + j125.66)10^{-6}\ \text{S} = 125.7 \times 10^{-6}\angle 88.63^\circ\ \text{S}$$

$$\text{Hence } Z_0 = \sqrt{\frac{45.29\angle 70.66^\circ}{125.7 \times 10^{-6}\angle 88.63^\circ}} = \sqrt{[0.360 \times 10^6\angle -17.97^\circ]}\ \Omega$$

i.e., characteristic impedance, $\mathbf{Z_0 = 600\angle -8.99^\circ\ \Omega}$

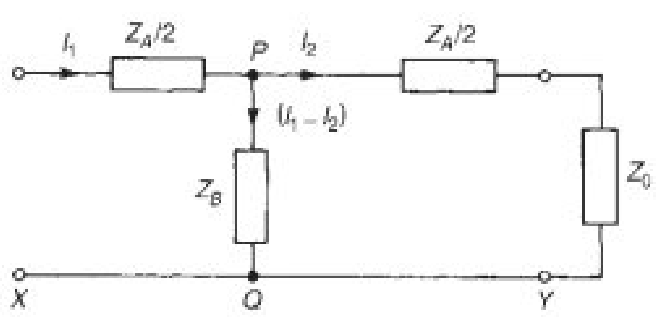


Figure 44.5

Propagation coefficient

Figure 44.5 shows a T-section with the series arm impedances each expressed as $Z_A/2$ ohms per unit length and the shunt impedance as Z_B ohms per unit length. The p.d. between points P and Q is given by:

$$V_{PQ} = (I_1 - I_2)Z_B = I_2\left(\frac{Z_A}{2} + Z_0\right)$$

i.e., $$I_1 Z_B - I_2 Z_B = \frac{I_2 Z_A}{2} + I_2 Z_0$$

Hence $$I_1 Z_B = I_2\left(Z_B + \frac{Z_A}{2} + Z_0\right)$$

from which $$\frac{I_1}{I_2} = \frac{Z_B + (Z_A/2) + Z_0}{Z_B}$$

From equation (41.1), page 760, $Z_0 = \sqrt{(Z_1{}^2 + 2Z_1Z_2)}$. In Figure 44.5, $Z_1 \equiv Z_A/2$ and $Z_2 \equiv Z_B$

Thus $$Z_0 = \sqrt{\left[\left(\frac{Z_A}{2}\right)^2 + 2\left(\frac{Z_A}{2}\right)Z_B\right]} = \sqrt{\left(\frac{Z_A{}^2}{4} + Z_AZ_B\right)}$$

Hence $$\frac{I_1}{I_2} = \frac{Z_B + (Z_A/2) + \sqrt{(Z_AZ_B + (Z_A{}^2/4))}}{Z_B}$$

$$= \frac{Z_B}{Z_B} + \frac{(Z_A/2)}{Z_B} + \frac{\sqrt{(Z_AZ_B + (Z_A{}^2/4))}}{Z_B}$$

$$= 1 + \frac{1}{2}\left(\frac{Z_A}{Z_B}\right) + \sqrt{\left(\frac{Z_AZ_B}{Z_B{}^2} + \frac{(Z_A{}^2/4)}{Z_B{}^2}\right)}$$

i.e., $$\frac{I_1}{I_2} = 1 + \frac{1}{2}\left(\frac{Z_A}{Z_B}\right) + \left[\frac{Z_A}{Z_B} + \frac{1}{4}\left(\frac{Z_A}{Z_B}\right)^2\right]^{1/2} \qquad (44.9)$$

From Section 44.4, $I_1/I_2 = e^\gamma$, where γ is the propagation coefficient. Also, from the binomial theorem:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \cdots$$

Thus $$\left[\frac{Z_A}{Z_B} + \frac{1}{4}\left(\frac{Z_A}{Z_B}\right)^2\right]^{1/2}$$

$$= \left(\frac{Z_A}{Z_B}\right)^{1/2} + \frac{1}{2}\left(\frac{Z_A}{Z_B}\right)^{-1/2}\frac{1}{4}\left(\frac{Z_A}{Z_B}\right)^2 + \cdots$$

Hence, from equation (44.9),

$$\frac{I_1}{I_2} = e^\gamma = 1 + \frac{1}{2}\left(\frac{Z_A}{Z_B}\right) + \left[\left(\frac{Z_A}{Z_B}\right)^{1/2} + \frac{1}{8}\left(\frac{Z_A}{Z_B}\right)^{3/2} + \cdots\right.$$

Rearranging gives: $e^\gamma = 1 + \left(\frac{Z_A}{Z_B}\right)^{1/2} + \frac{1}{2}\left(\frac{Z_A}{Z_B}\right) + \frac{1}{8}\left(\frac{Z_A}{Z_B}\right)^{3/2} + \cdots$

Let length XY in Figure 44.5 be a very short length of line δl and let impedance $Z_A = Z\delta l$, where $Z = R + j\omega L$ and $Z_B = 1/(Y\delta l)$, where $Y = G + j\omega C$

Then

$$e^{\gamma\delta l} = 1 + \left(\frac{Z\delta l}{1/Y\delta l}\right)^{1/2} + \frac{1}{2}\left(\frac{Z\delta l}{1/Y\delta l}\right) + \frac{1}{8}\left(\frac{Z\delta l}{1/Y\delta l}\right)^{3/2} + \cdots$$

$$= 1 + (ZY\delta l^2)^{1/2} + \frac{1}{2}(ZY\delta l^2) + \frac{1}{8}(ZY\delta l^2)^{3/2} + \cdots$$

$$= 1 + (ZY)^{1/2}\delta l + \frac{1}{2}(ZY)(\delta l)^2 + \frac{1}{8}(ZY)^{3/2}(\delta l)^3 + \cdots$$

$$= 1 + (ZY)^{1/2}\delta l,$$

if $(\delta l)^2$, $(\delta l)^3$ and higher powers are considered as negligible.

e^x may be expressed as a series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Comparison with $e^{\gamma\delta l} = 1 + (ZY)^{1/2}\delta l$ shows that $\gamma\delta l = (ZY)^{1/2}\delta l$ i.e., $\gamma = \sqrt{(ZY)}$. Thus

propagation coefficient, $$\boldsymbol{\gamma = \sqrt{[(R + j\omega L)(G + j\omega C)]}} \qquad (44.10)$$

The unit of γ is $\sqrt{(\Omega)(S)}$, i.e., $\sqrt{[(\Omega)(1/\Omega)]}$, thus γ is dimensionless, as expected, since $I_1/I_2 = e^{\gamma}$, from which $\gamma = \ln(I_1/I_2)$, i.e., a ratio of two currents. For a lossless line, $R = G = 0$ and

$$\boldsymbol{\gamma = \sqrt{(j\omega L)(j\omega C)} = j\omega\sqrt{(LC)}} \qquad (44.11)$$

Equations (44.7) and (44.10) are used to determine the characteristic impedance Z_0 and propagation coefficient γ of a transmission line in terms of the primary constants R, L, G and C. When $R = G = 0$, i.e., losses are neglected, equations (44.8) and (44.11) are used to determine Z_0 and γ.

Problem 8. A transmission line having negligible losses has primary line constants of inductance $L = 0.5$ mH/loop km and capacitance $C = 0.12$ µF/km. Determine, at an operating frequency of 400 kHz, (a) the characteristic impedance, (b) the propagation coefficient, (c) the wavelength on the line, and (d) the velocity of propagation, in metres per second, of a signal.

(a) Since the line is lossfree, from equation (44.8), the characteristic impedance Z_0 is given by

$$Z_0 = \sqrt{\frac{L}{C}} = \sqrt{\frac{0.5 \times 10^{-3}}{0.12 \times 10^{-6}}} = \mathbf{64.55\ \Omega}$$

(b) From equation (44.11), for a lossfree line, the propagation coefficient γ is given by

$$\gamma = j\omega\sqrt{(LC)} = j(2\pi 400 \times 10^3)\sqrt{[(0.5 \times 10^{-3})(0.12 \times 10^{-6})]}$$

$$= j19.47 \text{ or } \mathbf{0 + j19.47}$$

Since $\gamma = \alpha + j\beta$, the attenuation coefficient $\alpha = 0$ and the phase-shift coefficient, $\beta = 19.47$ rad/km.

(c) From equation (44.2), wavelength $\lambda = \dfrac{2\pi}{\beta} = \dfrac{2\pi}{19.47}$

$$= \mathbf{0.323\ km\ or\ 323\ m}$$

(d) From equation (44.3), velocity of propagation $u = f\lambda$

$$= (400 \times 10^3)(323) = \mathbf{129 \times 10^6\ m/s}.$$

> Problem 9. At a frequency of 1 kHz the primary constants of a transmission line are resistance $R = 25\ \Omega$/loop km, inductance $L = 5$ mH/loop km, capacitance $C = 0.04\ \mu$F/km and conductance $G = 80\ \mu$S/km. Determine for the line (a) the characteristic impedance, (b) the propagation coefficient, (c) the attenuation coefficient and (d) the phase-shift coefficient.

(a) From equation (44.7),

characteristic impedance $Z_0 = \sqrt{\dfrac{R + j\omega L}{G + j\omega C}}$ ohms

$$R + j\omega L = 25 + j(2\pi 1000)(5 \times 10^{-3}) = (25 + j31.42)$$

$$= 40.15\angle 51.49°\ \Omega$$

$$G + j\omega C = 80 \times 10^{-6} + j(2\pi 1000)(0.04 \times 10^{-6})$$

$$= (80 + j251.33)10^{-6} = 263.76 \times 10^{-6}\angle 72.34°\ \text{S}$$

Thus characteristic impedance

$$Z_0 = \sqrt{\frac{40.15\angle 51.49°}{263.76 \times 10^{-6}\angle 72.34°}} = \mathbf{390.2\angle{-10.43°}\ \Omega}$$

(b) From equation (44.10), propagation coefficient

$$\gamma = \sqrt{[(R + j\omega L)(G + j\omega C)]}$$

$$= \sqrt{[(40.15\angle 51.49°)(263.76 \times 10^{-6}\angle 72.34°)]}$$

$$= \sqrt{(0.01059\angle 123.83°)} = \mathbf{0.1029\angle 61.92°}$$

(c) $\gamma = \alpha + j\beta = 0.1029(\cos 61.92° + j \sin 61.92°)$,
i.e., $\gamma = 0.0484 + j0.0908$

Thus the attenuation coefficient, $\boldsymbol{\alpha} = \mathbf{0.0484\ nepers/km}$

(d) The phase shift coefficient, $\boldsymbol{\beta} = \mathbf{0.0908\ rad/km}$

Problem 10. An open wire line is 300 km long and is terminated in its characteristic impedance. At the sending end is a generator having an open-circuit e.m.f. of 10.0 V, an internal impedance of $(400 + j0)\Omega$ and a frequency of 1 kHz. If the line primary constants are $R = 8$ Ω/loop km, $L = 3$ mH/loop km, $C =$ 7500 pF/km and $G = 0.25$ μS/km, determine (a) the characteristic impedance, (b) the propagation coefficient, (c) the attenuation and phase-shift coefficients, (d) the sending-end current, (e) the receiving-end current, (f) the wavelength on the line, and (g) the speed of transmission of signal.

(a) From equation (44.7),

$$\text{characteristic impedance, } Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \text{ ohms}$$

$$R + j\omega L = 8 + j(2\pi 1000)(3 \times 10^{-3})$$
$$= 8 + j6\pi = 20.48\angle 67.0° \ \Omega$$

$$G + j\omega C = 0.25 \times 10^{-6} + j(2\pi 1000)(7500 \times 10^{-12})$$
$$= (0.25 + j47.12)10^{-6} = 47.12 \times 10^{-6}\angle 89.70° \text{ S}$$

Hence characteristic impedance

$$Z_0 = \sqrt{\frac{20.48\angle 67.0°}{47.12 \times 10^{-6}\angle 89.70°}} = \mathbf{659.3\angle -11.35° \ \Omega}$$

(b) From equation (44.10), propagation coefficient

$$\gamma = \sqrt{[(R + j\omega L)(G + j\omega C)]} =$$
$$\sqrt{[(20.48\angle 67.0°)(47.12 \times 10^{-6}\angle 89.70°)]} = \mathbf{0.03106\angle 78.35°}$$

(c) $\gamma = \alpha + j\beta = 0.03106(\cos 78.35° + j\sin 78.35°)$

$$= 0.00627 + j0.03042$$

Hence the attenuation coefficient, $\boldsymbol{\alpha} = \mathbf{0.00627}$ **Np/km** and the phase shift coefficient, $\boldsymbol{\beta} = \mathbf{0.03042}$ **rad/km**

(d) With reference to Figure 44.6, since the line is matched, i.e., terminated in its characteristic impedance, $V_S/I_S = Z_0$. Also

$$V_S = V_G - I_S Z_G = 10.0 - I_S(400 + j0)$$

Thus $I_S = \dfrac{V_S}{Z_0} = \dfrac{10.0 - 400 I_S}{Z_0}$

Rearranging gives: $I_S Z_0 = 10.0 - 400\, I_S$, from which,

$$I_S(Z_0 + 400) = 10.0$$

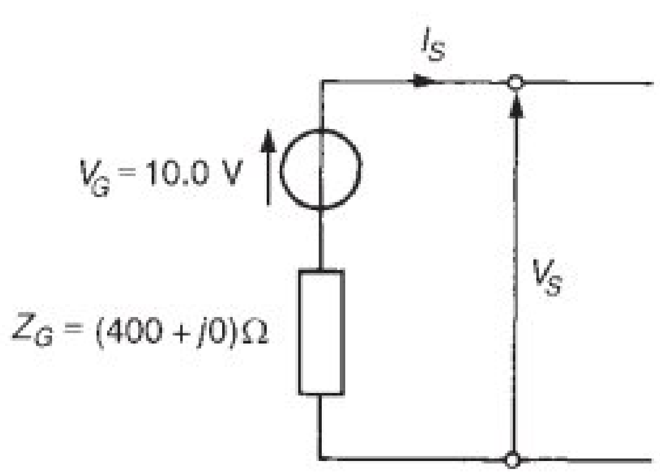


Figure 44.6

Thus the sending-end current,

$$I_S = \frac{10.0}{Z_0 + 400} = \frac{10.0}{659.3\angle{-11.35°} + 400}$$

$$= \frac{10.0}{646.41 - j129.75 + 400} = \frac{10.0}{1054.4\angle{-7.07°}}$$

$$= \mathbf{9.484\angle 7.07° \ mA}$$

(e) From equation (44.4), the receiving-end current,

$$I_R = I_S e^{-n\gamma} = I_S e^{-n\alpha} \angle{-n\beta}$$

$$= (9.484\angle 7.07°)e^{-(300)(0.00627)} \angle{-(300)(0.03042)}$$

$$= 9.484\angle 7.07° e^{-1.881} \angle{-9.13 \text{ rad}}$$

$$= 1.446\angle{-516°} \text{ mA} = \mathbf{1.446\angle{-156°} \ mA}$$

(f) From equation (44.2),

$$\text{wavelength, } \lambda = \frac{2\pi}{\beta} = \frac{2\pi}{0.03042} = \mathbf{206.5 \ km}$$

(g) From equation (44.3),

$$\text{speed of transmission, } u = f\lambda = (1000)(206.5)$$

$$= 206.5 \times 10^3 \text{ km/s} = \mathbf{206.5 \times 10^6 \ m/s}$$

Further problems on the characteristic impedance and the propagation coefficient in terms of the primary constants may be found in Section 44.9, problems 7 to 11, page 898.

44.6 Distortion on transmission lines

If the waveform at the receiving end of a transmission line is not the same shape as the waveform at the sending end, **distortion** is said to have occurred. The three main causes of distortion on transmission lines are as follows.

(i) The characteristic impedance Z_0 of a line varies with the operating frequency, i.e., from equation (44.7),

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}} \text{ ohms}$$

The terminating impedance of the line may not vary with frequency in the same manner.

In the above equation for Z_0, if the frequency is very low, ω is low and $Z_0 \approx \sqrt{(R/G)}$. If the frequency is very high, then $\omega L \gg R$,

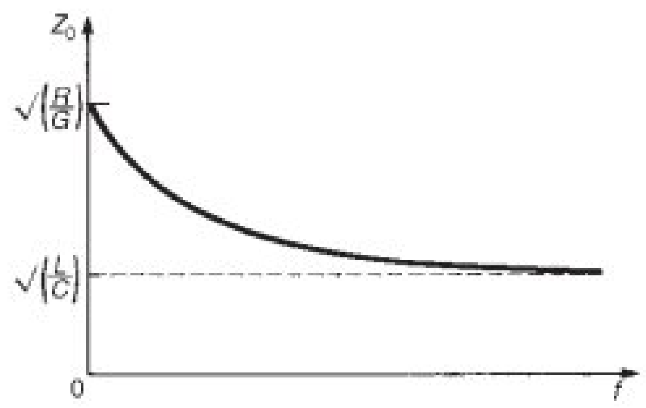


Figure 44.7

$\omega C \gg G$ and $Z_0 \approx \sqrt{(L/C)}$. A graph showing the variation of Z_0 with frequency f is shown in Figure 44.7.

If the characteristic impedance is to be constant throughout the entire operating frequency range then the following condition is required: $\sqrt{(L/C)} = \sqrt{(R/G)}$, i.e., $L/C = R/G$, from which

$$\boxed{LG = CR} \qquad (44.12)$$

Thus, in a transmission line, if $LG = CR$ it is possible to provide a termination equal to the characteristic impedance Z_0 at all frequencies.

(ii) The attenuation of a line varies with the operating frequency (since $\gamma = \sqrt{[(R + j\omega L)(G + j\omega C)]}$, from equation (44.10)), thus waves of differing frequencies and component frequencies of complex waves are attenuated by different amounts.

From the above equation for the propagation coefficient:

$$\gamma^2 = (R + j\omega L)(G + j\omega C)$$
$$= RG + j\omega(LG + CR) - \omega^2 LC$$

If $LG = CR = x$, then $LG + CR = 2x$ and $LG + CR$ may be written as $2\sqrt{x^2}$, i.e., $LG + CR$ may be written as $2\sqrt{[(LG)(CR)]}$.

Thus $\gamma^2 = RG + j\omega(2\sqrt{[(LG)(CR)]}) - \omega^2 LC$

$$= [\sqrt{(RG)} + j\omega\sqrt{(LC)}]^2$$

and $\gamma = \sqrt{(RG)} + j\omega\sqrt{(LC)}$

Since

$\gamma = \alpha + j\beta$, **attenuation coefficient,** $\boxed{\boldsymbol{\alpha = \sqrt{(RG)}}}$ (44.13)

and **phase shift coefficient,** $\boxed{\boldsymbol{\beta = \omega\sqrt{(LC)}}}$ (44.14)

Thus, in a transmission line, if $LG = CR$, $\alpha = \sqrt{(RG)}$, i.e., the attenuation coefficient is independent of frequency and all frequencies are equally attenuated.

(iii) The delay time, or the time of propagation, and thus the velocity of propagation, varies with frequency and therefore waves of different frequencies arrive at the termination with differing delays. From equation (44.14), the phase-shift coefficient, $\beta = \omega\sqrt{(LC)}$ when $LG = CR$.

$$\text{Velocity of propagation, } u = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{(LC)}} = \frac{1}{\sqrt{(LC)}} \qquad (44.15)$$

Thus, in a transmission line, if $LG = CR$, the velocity of propagation, and hence the time delay, is independent of frequency.

From the above it appears that the condition $LG = CR$ is appropriate for the design of a transmission line, since under this condition no distortion is introduced. This means that the signal at the receiving end is the same as the sending-end signal except that it is reduced in amplitude and delayed by a fixed time. Also, with no distortion, the attenuation on the line is a minimum. In practice, however, $R/L \gg G/C$. The inductance is usually low and the capacitance is large and not easily reduced. Thus if the condition $LG = CR$ is to be achieved in practice, either L or G must be increased since neither C or R can really be altered. It is undesirable to increase G since the attenuation and power losses increase. Thus the inductance L is the quantity that needs to be increased and such an artificial increase in the line inductance is called **loading**. This is achieved either by inserting inductance coils at intervals along the transmission line — this being called **'lumped loading'** — or by wrapping the conductors with a high-permeability metal tape — this being called **'continuous loading'**.

Problem 11. An underground cable has the following primary constants: resistance $R = 10\ \Omega$/loop km, inductance $L = 1.5$ mH/loop km, conductance $G = 1.2\ \mu$S/km and capacitance $C = 0.06\ \mu$F/km. Determine by how much the inductance should be increased to satisfy the condition for minimum distortion.

From equation (44.12), the condition for minimum distortion is given by $LG = CR$, from which,

$$\text{inductance } L = \frac{CR}{G} = \frac{(0.06 \times 10^{-6})(10)}{1.2 \times 10^{-6}} = 0.5 \text{ H or } 500 \text{ mH}$$

Thus the inductance should be increased by $(500 - 1.5)$ mH, i.e., **498.5 mH** per loop km, for minimum distortion.

Problem 12. A cable has the following primary constants: resistance $R = 80\ \Omega$/loop km, conductance, $G = 2\ \mu$S/km, and capacitance $C = 5$ nF/km. Determine, for minimum distortion at a frequency of 1.5 kHz (a) the value of inductance per loop kilometre required, (b) the propagation coefficient, (c) the velocity of propagation of signal, and (d) the wavelength on the line

(a) From equation (44.12), for minimum distortion, $LG = CR$, from which, inductance per loop kilometre,

$$L = \frac{CR}{G} = \frac{(5 \times 10^{-9})(80)}{(2 \times 10^{-6})} = \mathbf{0.20\ H\ or\ 200\ mH}$$

(b) From equation (44.13), attenuation coefficient,

$$\alpha = \sqrt{(RG)} = \sqrt{[(80)(2 \times 10^{-6})]} = 0.0126 \text{ Np/km}$$

and from equation (44.14), phase shift coefficient,

$$\beta = \omega\sqrt{(LC)} = (2\pi 1500)\sqrt{[(0.20)(5 \times 10^{-9})]} = 0.2980 \text{ rad/km}$$

Hence the propagation coefficient,

$$\gamma = \alpha + j\beta = \mathbf{(0.0126 + j0.2980)} \text{ or } \mathbf{0.2983\angle 87.58^\circ}$$

(c) From equation (44.15), velocity of propagation,

$$u = \frac{1}{\sqrt{(LC)}} = \frac{1}{\sqrt{[(0.2)(5 \times 10^{-9})]}}$$

$$= \mathbf{31\,620 \text{ km/s}} \text{ or } \mathbf{31.62 \times 10^6 \text{ m/s}}$$

(d) Wavelength, $\lambda = \dfrac{u}{f} = \dfrac{31.62 \times 10^6}{1500} \text{ m} = \mathbf{21.08 \text{ km}}$

Further problems on distortion on transmission lines may be found in Section 44.9, problems 12 and 13, page 899.

44.7 Wave reflection and the reflection coefficient

In earlier sections of this chapter it was assumed that the transmission line had been properly terminated in its characteristic impedance or regarded as an infinite line. In practice, of course, all lines have a definite length and often the terminating impedance does not have the same value as the characteristic impedance of the line. When this is the case, the transmission line is said to have a **'mismatched load'**.

The forward-travelling wave moving from the source to the load is called the **incident wave** or the sending-end wave. With a mismatched load the termination will absorb only a part of the energy of the incident wave, the remainder being forced to return back along the line toward the source. This latter wave is called the **reflected wave**.

Electrical energy is transmitted by a transmission line; when such energy arrives at a termination that has a value different from the characteristic impedance, it experiences a sudden change in the impedance of the medium. When this occurs, some reflection of incident energy occurs and the reflected energy is lost to the receiving load. (Reflections commonly occur in nature when a change of transmission medium occurs; for example, sound waves are reflected at a wall, which can produce echoes, and light rays are reflected by mirrors.)

If a transmission line is terminated in its characteristic impedance, no reflection occurs; if terminated in an open circuit or a short circuit, total reflection occurs, i.e., the whole of the incident wave reflects along the line. Between these extreme possibilities, all degrees of reflection are possible.

Open-circuited termination

If a length of transmission line is open-circuited at the termination, no current can flow in it and thus no power can be absorbed by the termination. This condition is achieved if a current is imagined to be reflected from the termination, the reflected current having the same magnitude as the incident wave but with a phase difference of 180°. Also, since no power is absorbed at the termination (it is all returned back along the line), the reflected voltage wave at the termination must be equal to the incident wave. Thus the voltage at the termination must be doubled by the open circuit. The resultant current (and voltage) at any point on the transmission line and at any instant of time is given by the sum of the currents (and voltages) due to the incident and reflected waves (see Section 44.8).

Short-circuit termination

If the termination of a transmission line is short-circuited, the impedance is zero, and hence the voltage developed across it must be zero. As with the open-circuit condition, no power is absorbed by the termination. To obtain zero voltage at the termination, the reflected voltage wave must be equal in amplitude but opposite in phase (i.e., 180° phase difference) to the incident wave. Since no power is absorbed, the reflected current wave at the termination must be equal to the incident current wave and thus the current at the end of the line must be doubled at the short circuit. As with the open-circuited case, the resultant voltage (and current) at any point on the line and at any instant of time is given by the sum of the voltages (and currents) due to the incident and reflected waves.

Energy associated with a travelling wave

A travelling wave on a transmission line may be thought of as being made up of electric and magnetic components. Energy is stored in the magnetic field due to the current (energy $= \frac{1}{2}LI^2$ — see page 751) and energy is stored in the electric field due to the voltage (energy $= \frac{1}{2}CV^2$ — see page 738). It is the continual interchange of energy between the magnetic and electric fields, and *vice versa*, that causes the transmission of the total electromagnetic energy along the transmission line.

When a wave reaches an open-circuited termination the magnetic field collapses since the current I is zero. Energy cannot be lost, but it can change form. In this case it is converted into electrical energy, adding to that already caused by the existing electric field. The voltage at the termination consequently doubles and this increased voltage starts the movement of a reflected wave back along the line. A magnetic field will be set up by this movement and the total energy of the reflected wave will again be shared between the magnetic and electric field components.

When a wave meets a short-circuited termination, the electric field collapses and its energy changes form to the magnetic energy. This results in a doubling of the current.

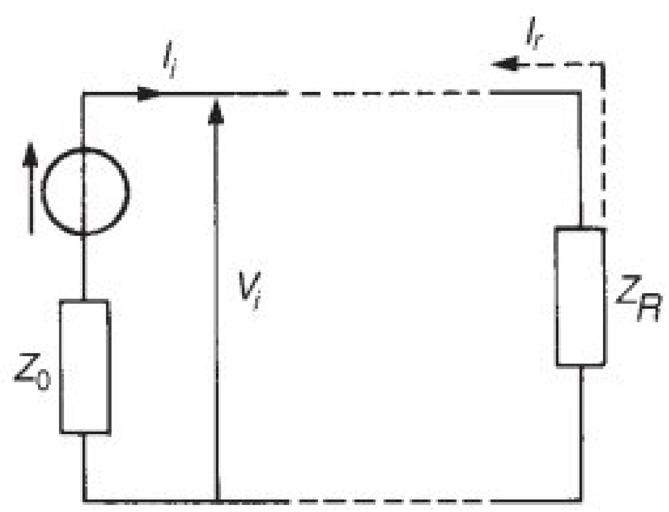


Figure 44.8

Reflection coefficient

Let a generator having impedance Z_0 (this being equal to the characteristic impedance of the line) be connected to the input terminals of a transmission line which is terminated in an impedance Z_R, where $Z_0 \neq Z_R$, as shown in Figure 44.8. The sending-end or incident current I_i flowing from the source generator flows along the line and, until it arrives at the termination Z_R behaves as though the line were infinitely long or properly terminated in its characteristic impedance, Z_0.

The incident voltage V_i shown in Figure 44.8 is given by:

$$V_i = I_i Z_0 \qquad (44.12)$$

from which, $$I_i = \frac{V_i}{Z_0} \qquad (44.13)$$

At the termination, the conditions must be such that:

$$Z_R = \frac{\text{total voltage}}{\text{total current}}$$

Since $Z_R \neq Z_0$, part of the incident wave will be reflected back along the line from the load to the source. Let the reflected voltage be V_r and the reflected current be I_r. Then

$$V_r = -I_r Z_0 \qquad (44.14)$$

from which, $$I_r = -\frac{V_r}{Z_0} \qquad (44.15)$$

(Note the minus sign, since the reflected voltage and current waveforms travel in the opposite direction to the incident waveforms.)

Thus, at the termination,

$$Z_R = \frac{\text{total voltage}}{\text{total current}} = \frac{V_i + V_r}{I_i + I_r}$$

$$= \frac{I_i Z_0 - I_r Z_0}{I_i + I_r} \text{ from equations (44.12) and (44.14)}$$

i.e., $$Z_R = \frac{Z_0(I_i - I_r)}{(I_i + I_r)}$$

Hence $$Z_R(I_i + I_r) = Z_0(I_i - I_r)$$

$$Z_R I_i + Z_R I_r = Z_0 I_i - Z_0 I_r$$

$$Z_0 I_r + Z_R I_r = Z_0 I_i - Z_R I_i$$

$$I_r(Z_0 + Z_R) = I_i(Z_0 - Z_R)$$

from which $$\frac{I_r}{I_i} = \frac{Z_0 - Z_r}{Z_0 + Z_r}$$

The ratio of the reflected current to the incident current is called the **reflection coefficient** and is often given the symbol ρ, i.e.,

$$\boxed{\frac{I_r}{I_i} = \rho = \frac{Z_0 - Z_R}{Z_0 + Z_R}} \qquad (44.16)$$

By similar reasoning to above an expression for the ratio of the reflected to the incident voltage may be obtained. From above,

$$Z_R = \frac{V_i + V_r}{I_i + I_r} = \frac{V_i + V_r}{(V_i/Z_0) - (V_r/Z_0)}$$

from equations (44.13) and (44.15),

i.e., $Z_R = \dfrac{V_i + V_r}{(V_i - V_r)/Z_0}$

Hence $\dfrac{Z_R}{Z_0}(V_i - V_r) = V_i + V_r$

from which, $\dfrac{Z_R}{Z_0}V_i - \dfrac{Z_R}{Z_0}V_r = V_i + V_r$

Then $\dfrac{Z_R}{Z_0}V_i - V_i = V_r + \dfrac{Z_R}{Z_0}V_r$

and $V_i\left(\dfrac{Z_R}{Z_0} - 1\right) = V_r\left(1 + \dfrac{Z_R}{Z_0}\right)$

Hence $V_i\left(\dfrac{Z_R - Z_0}{Z_0}\right) = V_r\left(\dfrac{Z_0 + Z_R}{Z_0}\right)$

from which $\dfrac{V_r}{V_i} = \dfrac{Z_R - Z_0}{Z_0 + Z_R} = -\left(\dfrac{Z_0 - Z_R}{Z_0 + Z_R}\right)$ (44.17)

Hence $\boxed{\dfrac{V_r}{V_i} = -\dfrac{I_r}{I_i} = -\rho}$ (44.18)

Thus the ratio of the reflected to the incident voltage has the same magnitude as the ratio of reflected to incident current, but is of opposite sign. From equations (44.16) and (44.17) it is seen that when $Z_R = Z_0$, $\rho = 0$ and there is no reflection.

> Problem 13. A cable which has a characteristic impedance of 75 Ω is terminated in a 250 Ω resistive load. Assuming that the cable has negligible losses and the voltage measured across the terminating load is 10 V, calculate the value of (a) the reflection coefficient for the line, (b) the incident current, (c) the incident voltage, (d) the reflected current, and (e) the reflected voltage.

(a) From equation (44.16),

$$\textbf{reflection coefficient, } \rho = \frac{Z_0 - Z_R}{Z_0 + Z_R} = \frac{75 - 250}{75 + 250} = \frac{-175}{325} = \mathbf{-0.538}$$

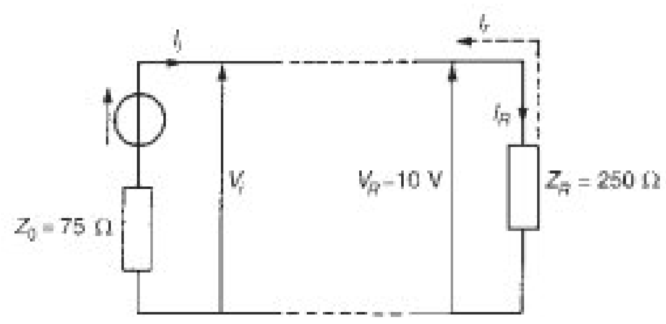


Figure 44.9

(b) The circuit diagram is shown in Figure 44.9. Current flowing in the terminating load,

$$I_R = \frac{V_R}{Z_R} = \frac{10}{250} = 0.04 \text{ A}$$

However, current $I_R = I_i + I_r$. From equation (44.16), $I_r = \rho I_i$

Thus $I_R = I_i + \rho I_i = I_i(1 + \rho)$

from which **incident current,** $\boldsymbol{I_i} = \dfrac{I_R}{(1+\rho)}$

$$= \frac{0.04}{1 + (-0.538)} = \mathbf{0.0866\ A} \text{ or } \mathbf{86.6\ mA}$$

(c) From equation (44.12),

incident voltage, $\boldsymbol{V_i} = I_i Z_0 = (0.0866)(75) = \mathbf{6.50\ V}$

(d) Since $I_R = I_i + I_r$

reflected current, $\boldsymbol{I_r} = I_R - I_i = 0.04 - 0.0866$

$$= \mathbf{-0.0466\ A} \text{ or } \mathbf{-46.6\ mA}$$

(e) From equation (44.14),

reflected voltage, $\boldsymbol{V_r} = -I_r Z_0 = -(-0.0466)(75) = \mathbf{3.50\ V}$

> Problem 14. A long transmission line has a characteristic impedance of $(500 - j40)\Omega$ and is terminated in an impedance of (a) $(500 + j40)\Omega$ and (b) $(600 + j20)\Omega$. Determine the magnitude of the reflection coefficient in each case.

(a) From equation (44.16), reflection coefficient,

$$\rho = \frac{Z_0 - Z_R}{Z_0 + Z_R}$$

When $Z_0 = (500 - j40)\Omega$ and $Z_R = (500 + j40)\Omega$

$$\rho = \frac{(500 - j40) - (500 + j40)}{(500 - j40) + (500 + j40)} = \frac{-j80}{1000} = -j0.08$$

Hence the magnitude of the reflection coefficient, $|\rho| = \mathbf{0.08}$

(b) When $Z_0 = (500 - j40)\Omega$ and $Z_R = (600 + j20)\Omega$

$$\rho = \frac{(500 - j40) - (600 + j20)}{(500 - j40) + (600 + j20)} = \frac{-100 - j60}{1100 - j20}$$

$$= \frac{116.62\angle - 149.04^\circ}{1100.18\angle{-1.04^\circ}}$$

$$= 0.106\angle{-148^\circ}$$

Hence the magnitude of the reflection coefficient, $|\rho| = \mathbf{0.106}$

> Problem 15. A loss-free transmission line has a characteristic impedance of $500\angle 0^\circ\ \Omega$ and is connected to an aerial of impedance $(320 + j240)\Omega$. Determine (a) the magnitude of the ratio of the reflected to the incident voltage wave, and (b) the incident voltage if the reflected voltage is $20\angle 35^\circ$ V

(a) From equation (44.17), the ratio of the reflected to the incident voltage is given by:

$$\frac{V_r}{V_i} = \frac{Z_R - Z_0}{Z_R + Z_0}$$

where Z_0 is the characteristic impedance $500\angle 0^\circ\ \Omega$ and Z_R is the terminating impedance $(320 + j240)\Omega$.

$$\text{Thus} \quad \frac{V_r}{V_i} = \frac{(320 + j240) - 500\angle 0^\circ}{500\angle 0^\circ + (320 + j240)} = \frac{-180 + j240}{820 + j240}$$

$$= \frac{300\angle 126.87^\circ}{854.4\angle 16.31^\circ} = 0.351\angle 110.56^\circ$$

Hence the magnitude of the ratio $V_r : V_i$ is **0.351**

(b) Since $V_r/V_i = 0.351\angle 110.56^\circ$,

$$\text{incident voltage, } V_i = \frac{V_r}{0.351\angle 110.56^\circ}$$

Thus, when $V_r = 20\angle 35^\circ$ V,

$$V_i = \frac{20\angle 35^\circ}{0.351\angle 110.56^\circ} = \mathbf{57.0\angle{-75.56^\circ}\ V}$$

Further problems on the reflection coefficient may be found in Section 44.9, problems 14 to 16, page 899.

44.8 Standing waves and the standing wave ratio

Consider a lossfree transmission line **open-circuited** at its termination. An incident current waveform is completely reflected at the termination, and, as stated in Section 44.7, the reflected current is of the same magnitude as the incident current but is 180° out of phase. Figure 44.10(a) shows the incident and reflected current waveforms drawn separately (shown as

I_i moving to the right and I_r moving to the left respectively) at a time $t = 0$, with $I_i = 0$ and decreasing at the termination.

The resultant of the two waves is obtained by adding them at intervals. In this case the resultant is seen to be zero. Figures 44.10(b) and (c) show the incident and reflected waves drawn separately as times $t = T/8$ seconds and $t = T/4$, where T is the periodic time of the signal. Again, the resultant is obtained by adding the incident and reflected waveforms at intervals. Figures 44.10(d) to (h) show the incident and reflected

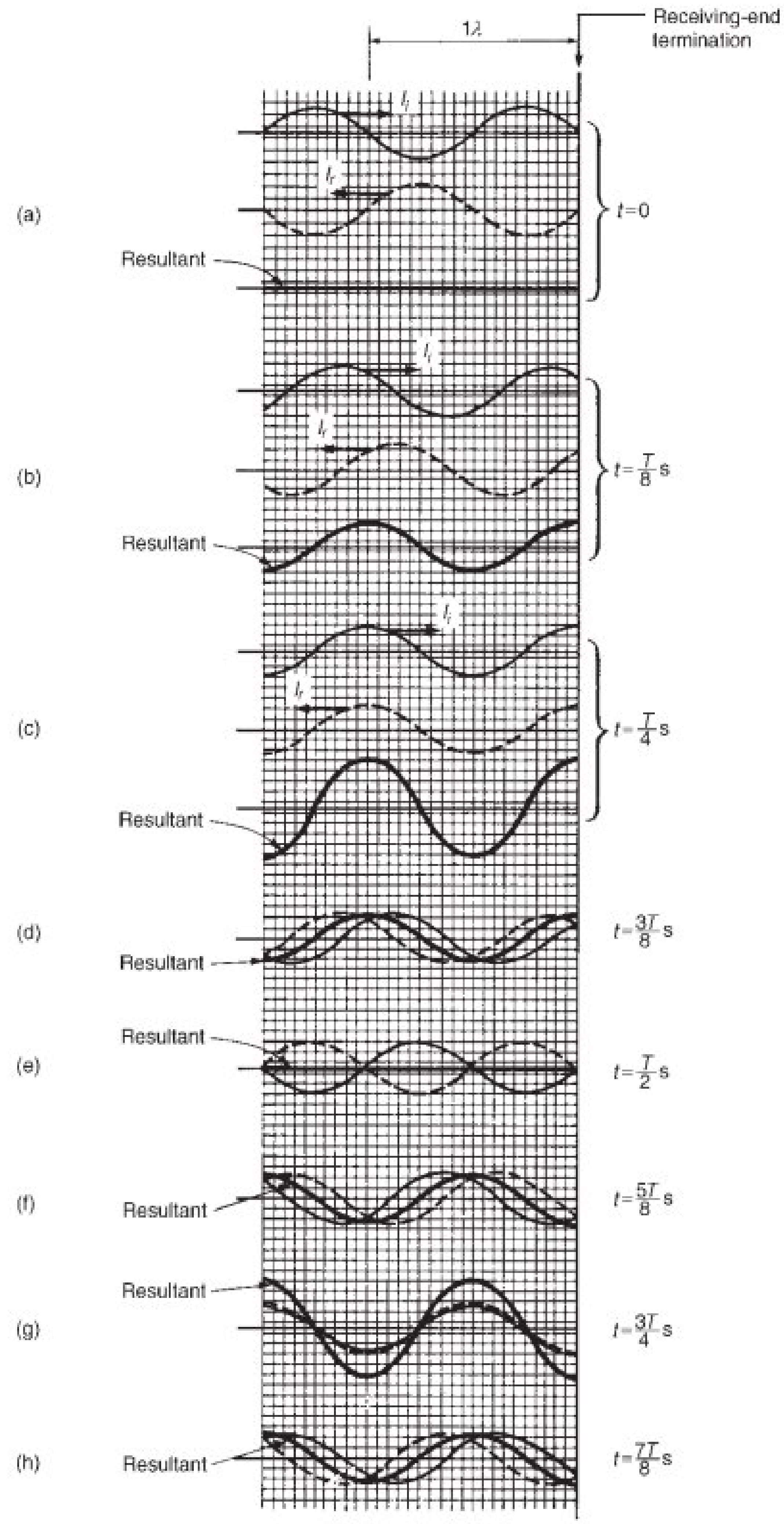


Figure 44.10 *Current waveforms on an open-circuited transmission line*

current waveforms plotted on the same axis, together with their resultant waveform, at times $t = 3T/8$ to $t = 7T/8$ at intervals of $T/8$.

If the resultant waveforms shown in Figures 44.10(a) to (g) are superimposed one upon the other, Figure 44.11 results. (Note that the scale has been increased for clarity.) The waveforms show clearly that waveform (a) moves to (b) after $T/8$, then to (c) after a further period of $T/8$, then to (d), (e), (f), (g) and (h) at intervals of $T/8$. It is noted that at any particular point the current varies sinusoidally with time, but the amplitude of oscillation is different at different points on the line.

Whenever two waves of the same frequency and amplitude travelling in opposite directions are superimposed on each other as above, interference takes place between the two waves and a **standing** or **stationary wave** is produced. The points at which the current is always zero are called **nodes** (labelled N in Figure 44.11). The standing wave does not progress to the left or right and the nodes do not oscillate. Those points on the wave that undergo maximum disturbance are called **antinodes** (labelled A in Figure 44.11). The distance between adjacent nodes or adjacent antinodes is $\lambda/2$, where λ is the wavelength. A standing wave is therefore seen to be a periodic variation in the vertical plane taking place on the transmission line without travel in either direction.

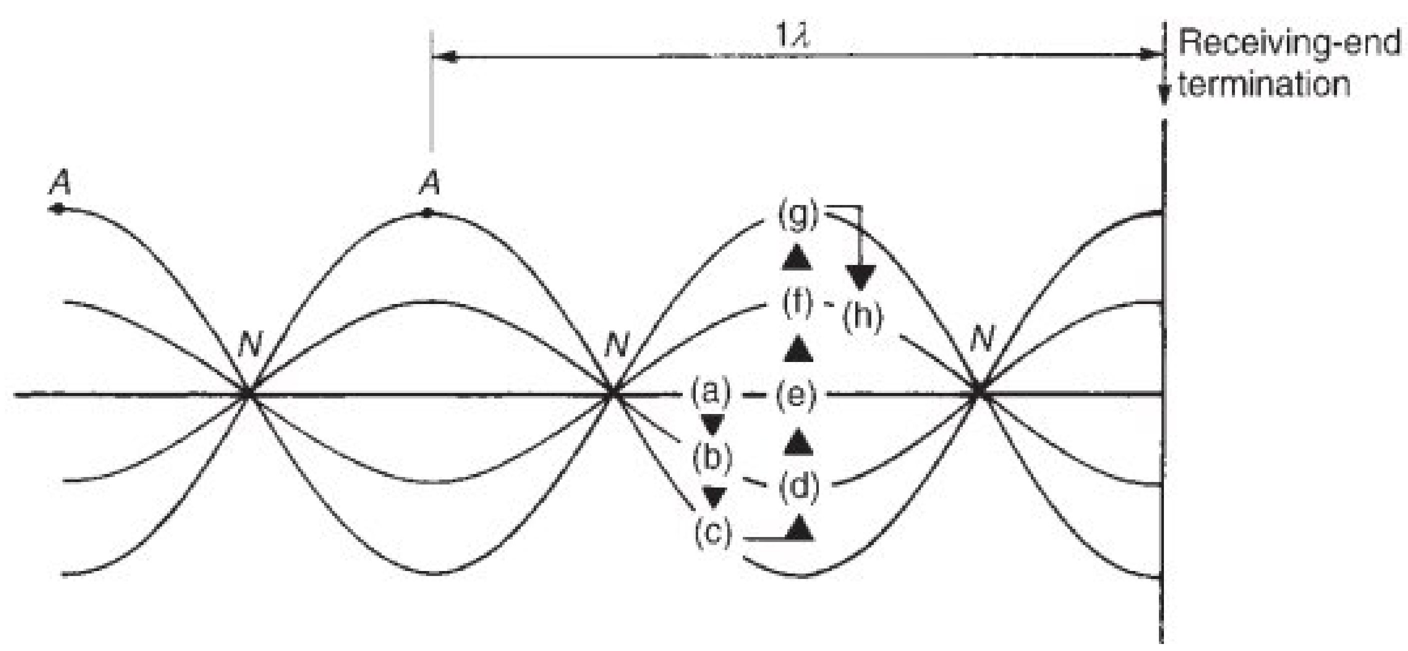


Figure 44.11

The resultant of the incident and reflected voltage for the open-circuit termination may be deduced in a similar manner to that for current. However, as stated in Section 44.7, when the incident voltage wave reaches the termination it is reflected without phase change. Figure 44.12 shows the resultant waveforms of incident and reflected voltages at intervals of $t = T/8$. Figure 44.13 shows all the resultant waveforms of Figure 44.12(a) to (h) superimposed; again, standing waves are seen to result. Nodes (labelled N) and antinodes (labelled A) are shown in Figure 44.13 and, in comparison with the current waves, are seen to occur 90° out of phase.

If the transmission line is short-circuited at the termination, it is the incident current that is reflected without phase change and the incident voltage that is reflected with a phase change of 180°. Thus the diagrams shown in Figures 44.10 and 44.11 representing current at an

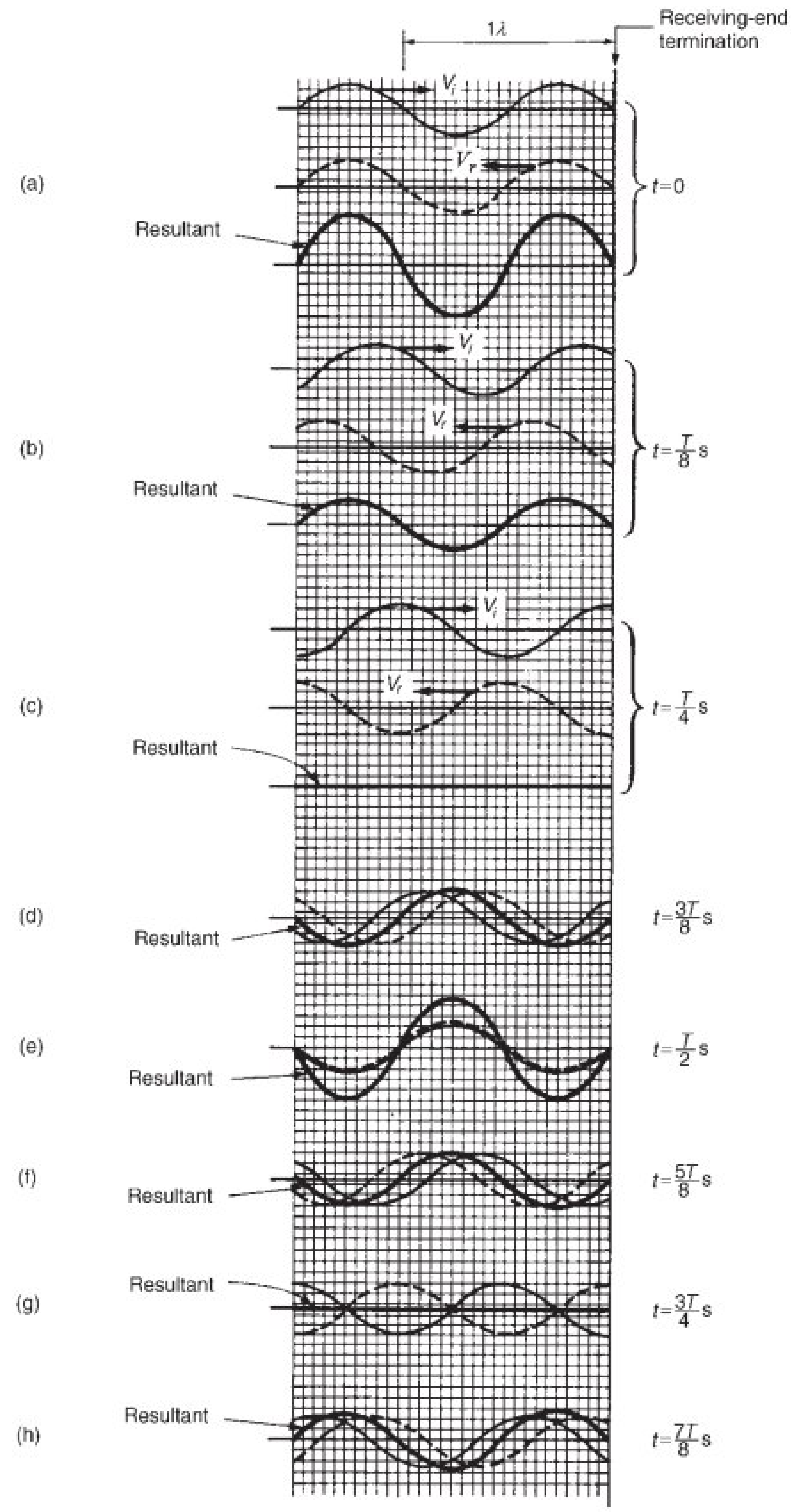


Figure 44.12 *Voltage waveforms on an open-circuited transmission line*

open-circuited termination may be used to represent voltage conditions at a short-circuited termination and the diagrams shown in Figures 44.12 and 44.13 representing voltage at an open-circuited termination may be used to represent current conditions at a short-circuited termination.

Figure 44.14 shows the rms current and voltage waveforms plotted on the same axis against distance for the case of total reflection, deduced

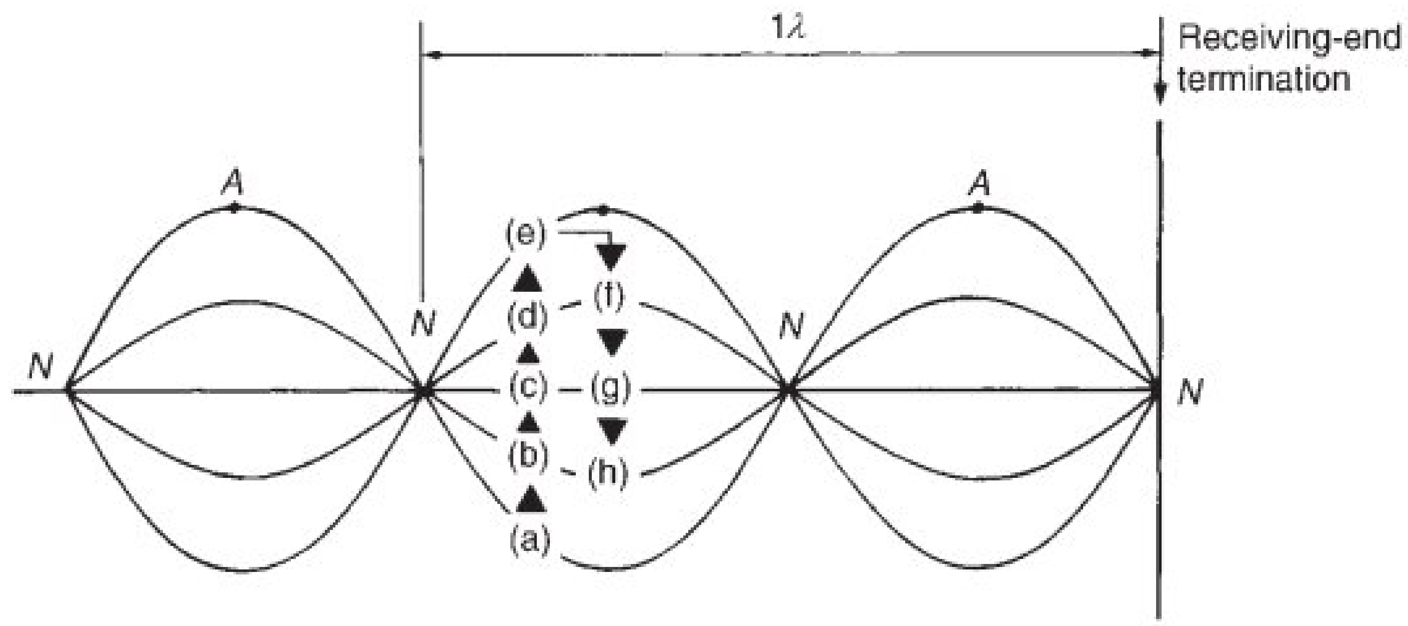


Figure 44.13

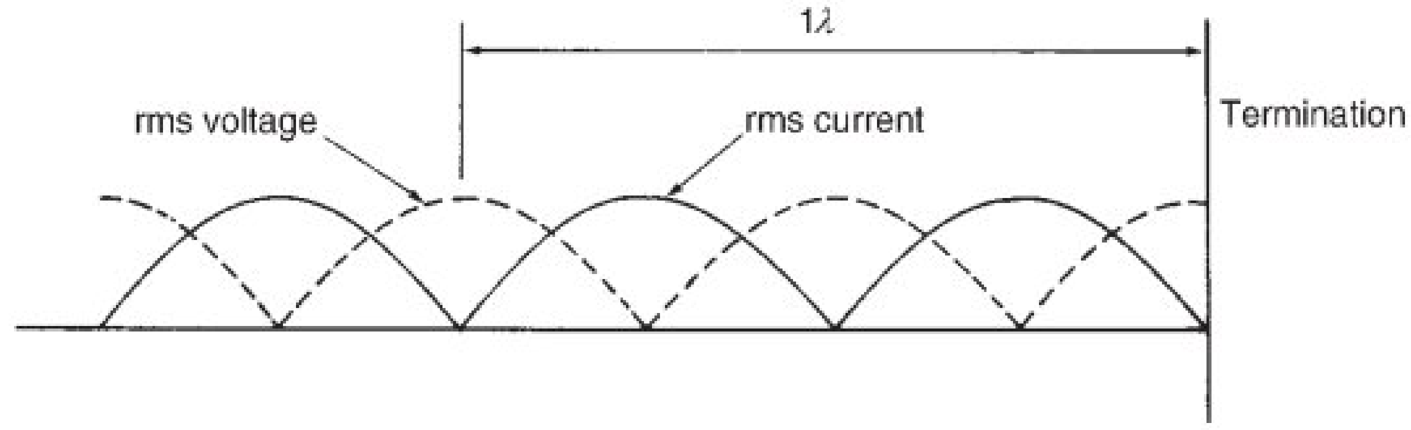


Figure 44.14

from Figures 44.11 and 44.13. The rms values are equal to the amplitudes of the waveforms shown in Figures 44.11 and 44.13, except that they are each divided by $\sqrt{2}$ (since, for a sine wave, rms value $= (1/\sqrt{2}) \times$ maximum value). With total reflection, the standing-wave patterns of rms voltage and current consist of a succession of positive sine waves with the voltage node located at the current antinode and the current node located at the voltage antinode. The termination is a current nodal point. The rms values of current and voltage may be recorded on a suitable rms instrument moving along the line. Such measurements of the maximum and minimum voltage and current can provide a reasonably accurate indication of the wavelength, and also provide information regarding the amount of reflected energy relative to the incident energy that is absorbed at the termination, as shown below.

Standing-wave ratio

Let the incident current flowing from the source of a mismatched low-loss transmission line be I_i and the current reflected at the termination be I_r. If I_{MAX} is the sum of the incident and reflected current, and I_{MIN} is their difference, then the **standing-wave ratio** (symbol **s**) on the line is defined as:

$$s = \frac{I_{MAX}}{I_{MIN}} = \frac{I_i + I_r}{I_i - I_r} \qquad (44.19)$$

Hence $s(I_i - I_r) = I_i + I_r$

$$sI_i - sI_r = I_i + I_r$$

$$sI_i - I_i = sI_r + I_r$$

$$I_i(s-1) = I_r(s+1)$$

i.e.,
$$\boxed{\frac{I_r}{I_i} = \left(\frac{s-1}{s+1}\right)} \tag{44.20}$$

The power absorbed in the termination $P_t = I_i{}^2 Z_0$ and the reflected power, $P_r = I_r{}^2 Z_0$. Thus $\dfrac{P_r}{P_t} = \dfrac{I_r{}^2 Z_0}{I_i{}^2 Z_0} = \left(\dfrac{I_r}{I_i}\right)^2$

Hence, from equation (44.20),

$$\boxed{\frac{P_r}{P_t} = \left(\frac{s-1}{s+1}\right)^2} \tag{44.21}$$

Thus the ratio of the reflected to the transmitted power may be calculated directly from the standing-wave ratio, which may be calculated from measurements of I_{MAX} and I_{MIN}. When a transmission line is properly terminated there is no reflection, i.e., $I_r = 0$, and from equation (44.19) the standing-wave ratio is 1. From equation (44.21), when $s = 1$, $P_r = 0$, i.e., there is no reflected power. In practice, the standing-wave ratio is kept as close to unity as possible.

From equation (44.16), the reflection coefficient, $\rho = I_r/I_i$ Thus, from equation (44.20), $|\rho| = \dfrac{s-1}{s+1}$

Rearranging gives: $|\rho|(s+1) = (s-1)$

$$|\rho|s + |\rho| = s - 1$$

$$1 + |\rho| = s(1 - |\rho|)$$

from which
$$\boxed{s = \frac{1+|\rho|}{1-|\rho|}} \tag{44.22}$$

Equation (44.22) gives an expression for the standing-wave ratio in terms of the magnitude of the reflection coefficient.

Problem 16. A transmission line has a characteristic impedance of $600\angle 0°\ \Omega$ and negligible loss. If the terminating impedance of the line is $(400 + j250)\Omega$, determine (a) the reflection coefficient and (b) the standing-wave ratio.

(a) From equation (44.16),

$$\text{reflection coefficient, } \rho = \frac{Z_0 - Z_R}{Z_0 + Z_R} = \frac{600\angle 0^\circ - (400 + j250)}{600\angle 0^\circ + (400 + j250)}$$

$$= \frac{200 - j250}{1000 + j250} = \frac{320.16\angle -51.34^\circ}{1030.78\angle 14.04^\circ}$$

Hence $\boldsymbol{\rho = 0.3106\angle - 65.38^\circ}$

(b) From above, $|\rho| = 0.3106$. Thus from equation (44.22),

$$\text{standing-wave ratio, } s = \frac{1 + |\rho|}{1 - |\rho|} = \frac{1 + 0.3106}{1 - 0.3106} = \mathbf{1.901}$$

> Problem 17. A low-loss transmission line has a mismatched load such that the reflection coefficient at the termination is $0.2\angle -120^\circ$. The characteristic impedance of the line is 80 Ω. Calculate (a) the standing-wave ratio, (b) the load impedance, and (c) the incident current flowing if the reflected current is 10 mA.

(a) From equation (44.22),

$$\textbf{standing-wave ratio, } s = \frac{1 + |\rho|}{1 - |\rho|} = \frac{1 + 0.2}{1 - 0.2} = \frac{1.2}{0.8} = \mathbf{1.5}$$

(b) From equation (44.16) reflection coefficient, $\rho = \dfrac{Z_0 - Z_R}{Z_0 + Z_R}$

Rearranging gives: $\rho(Z_0 + Z_R) = Z_0 - Z_R$,

from which $Z_R(\rho + 1) = Z_0(1 - \rho)$

and
$$\frac{Z_R}{Z_0} = \frac{1 - \rho}{1 + \rho} = \frac{1 - 0.2\angle -120^\circ}{1 + 0.2\angle -120^\circ} = \frac{1 - (-0.10 - j0.173)}{1 + (-0.10 - j0.173)}$$

$$= \frac{1.10 + j0.173}{0.90 - j0.173} = \frac{1.1135\angle 8.94^\circ}{0.9165\angle -10.88^\circ}$$

$$= 1.215\angle 19.82^\circ$$

Hence load impedance $Z_R = Z_0(1.215\angle 19.82^\circ) = (80)(1.215\angle 19.82^\circ)$

$$= \mathbf{97.2\angle 19.82^\circ\ \Omega} \text{ or } \mathbf{(91.4 + j33.0)\ \Omega}$$

(c) From equation (44.20),

$$\frac{I_r}{I_i} = \frac{s - 1}{s + 1}$$

Hence $\dfrac{10}{I_i} = \dfrac{1.5 - 1}{1.5 + 1} = \dfrac{0.5}{2.5} = 0.2$

Thus the **incident current,** $I_i = 10/0.2 =$ **50 mA**

Problem 18. The standing-wave ratio on a mismatched line is calculated as 1.60. If the incident power arriving at the termination is 200 mW, determine the value of the reflected power.

From equation (44.21),

$$\frac{P_r}{P_t} = \left(\frac{s-1}{s+1}\right)^2 = \left(\frac{1.60-1}{1.60+1}\right)^2 = \left(\frac{0.60}{2.60}\right)^2 = 0.0533$$

Hence the **reflected power,** $\boldsymbol{P_r} = 0.0533 P_t = (0.0533)(200)$

$= \mathbf{10.66\ mW}$

Further problems on the standing wave ratio may be found in Section 44.9 following, problems 17 to 21, page 899.

44.9 Further problems on transmission lines

Phase delay, wavelength and velocity of propagation

1 A parallel-wire air-spaced line has a phase-shift of 0.03 rad/km. Determine (a) the wavelength on the line, and (b) the speed of transmission of a signal of frequency 1.2 kHz.
[(a) 209.4 km (b) 251.3×10^6 m/s]

2 A transmission line has an inductance of 5 μH/m and a capacitance of 3.49 pF/m. Determine, for an operating frequency of 5 kHz, (a) the phase delay, (b) the wavelength on the line and (c) the velocity of propagation of the signal in metres per second.
[(a) 0.131 rad/km (b) 48 km (c) 240×10^6 m/s]

3 An air-spaced transmission line has a capacitance of 6.0 pF/m and the velocity of propagation of a signal is 225×10^6 m/s. If the operating frequency is 20 kHz, determine (a) the inductance per metre, (b) the phase delay, and (c) the wavelength on the line.
[(a) 3.29 μH/m (b) 0.558×10^{-3} rad/m (c) 11.25 km]

Current and voltage relationships

4 When the working frequency of a cable is 1.35 kHz, its attenuation is 0.40 Np/km and its phase-shift is 0.25 rad/km. The sending-end voltage and current are 8.0 V rms and 10.0 mA rms. Determine the voltage and current at a point 25 km down the line, assuming that the termination is equal to the characteristic impedance of the line.
[$V_R = 0.363\angle -6.25$ mV or $0.363\angle 1.90^\circ$ mV
$I_R = 0.454\angle -6.25$ μA or $0.454\angle 1.90^\circ$ μA]

5 A transmission line 8 km long has a characteristic impedance $600\angle -30^\circ$ Ω. At a particular frequency the attenuation coefficient of the line is 0.4 Np/km and the phase-shift coefficient is 0.20 rad/km.

Determine the magnitude and phase of the current at the receiving end if the sending-end voltage is $5\angle 0°$ V rms. [$0.340\angle -61.67$ mA]

6 The voltages at the input and at the output of a transmission line properly terminated in its characteristic impedance are 10 V and 4 V rms respectively. Determine the output voltage if the length of the line is trebled. [0.64 V]

Characteristic impedance and propagation constant

7 At a frequency of 800 Hz, the open-circuit impedance of a length of transmission line is measured as $500\angle -35°$ Ω and the short-circuit impedance as $300\angle -15°$ Ω. Determine the characteristic impedance of the line at this frequency. [$387.3\angle -25°$ Ω]

8 A transmission line has the following primary constants per loop kilometre run: $R = 12$ Ω, $L = 3$ mH, $G = 4$ μS and $C = 0.02$ μF. Determine the characteristic impedance of the line when the frequency is 750 Hz. [$443.3\angle -18.95°$ Ω]

9 A transmission line having negligible losses has primary constants: inductance $L = 1.0$ mH/loop km and capacitance $C = 0.20$ μF/km. Determine, at an operating frequency of 50 kHz, (a) the characteristic impedance, (b) the propagation coefficient, (c) the attenuation and phase-shift coefficients, (d) the wavelength on the line, and (e) the velocity of propagation of signal in metres per second.

[(a) 70.71 Ω (b) $j4.443$ (c) 0; 4.443 rad/km
(d) 1.414 km (e) 70.71×10^6 m/s]

10 At a frequency of 5 kHz the primary constants of a transmission line are: resistance $R = 12$ Ω/loop km, inductance $L = 0.50$ mH/loop km, capacitance $C = 0.01$ μF/km and $G = 60$ μS/km. Determine for the line (a) the characteristic impedance, (b) the propagation coefficient, (c) the attenuation coefficient, and (d) the phase-shift coefficient.

[(a) $248.6\angle -13.29°$ Ω (b) $0.0795\angle 65.91°$
(c) 0.0324 Np/km (d) 0.0726 rad/km]

11 A transmission line is 50 km in length and is terminated in its characteristic impedance. At the sending end a signal emanates from a generator which has an open-circuit e.m.f. of 20.0 V, an internal impedance of $(250 + j0)$Ω at a frequency of 1592 Hz. If the line primary constants are $R = 30$ Ω/loop km, $L = 4.0$ mH/loop km, $G = 5.0$ μS/km, and $C = 0.01$ μF/km, determine (a) the value of the characteristic impedance, (b) the propagation coefficient, (c) the attenuation and phase-shift coefficients, (d) the sending-end current, (e) the receiving-end current, (f) the wavelength on the line, and (g) the speed of transmission of a signal, in metres per second.

[(a) $706.6\angle -17°$ Ω (b) $0.0708\angle 70.14°$
(c) 0.024 Np/km; 0.067 rad/km
(d) $21.1\angle 12.58°$ mA (e) $6.35\angle -178.21°$ mA
(f) 94.34 km (g) 150.2×10^6 m/s]

Distortion on transmission lines

12 A cable has the following primary constants: resistance $R = 90\ \Omega$/loop km, inductance $L = 2.0$ mH/loop km, capacitance $C = 0.05\ \mu$F/km and conductance $G = 3.0\ \mu$S/km. Determine the value to which the inductance should be increased to satisfy the condition for minimum distortion. [1.5 H]

13 A condition of minimum distortion is required for a cable. Its primary constants are: $R = 40\ \Omega$/loop km, $L = 2.0$ mH/loop km, $G = 2.0\ \mu$S/km and $C = 0.08\ \mu$F/km. At a frequency of 100 Hz determine (a) the increase in inductance required, (b) the propagation coefficient, (c) the speed of signal transmission and (d) the wavelength on the line.
[(a) 1.598 H (b) $(8.944 + j225)10^{-3}$
(c) 2.795×10^6 m/s (d) 27.93 km]

Reflection coefficient

14 A coaxial line has a characteristic impedance of 100 Ω and is terminated in a 400 Ω resistive load. The voltage measured across the termination is 15 V. The cable is assumed to have negligible losses. Calculate for the line the values of (a) the reflection coefficient, (b) the incident current, (c) the incident voltage, (d) the reflected current, and (e) the reflected voltage.
[(a) −0.60 (b) 93.75 mA (c) 9.375 V
(d) −56.25 mA (e) 5.625 V]

15 A long transmission line has a characteristic impedance of $(400 - j50)\Omega$ and is terminated in an impedance of (i) $(400 + j50)\Omega$, (ii) $(500 + j60)\Omega$ and (iii) $400\angle 0°\ \Omega$. Determine the magnitude of the reflection coefficient in each case.
[(i) 0.125 (ii) 0.165 (iii) 0.062]

16 A transmission line which is loss-free has a characteristic impedance of $600\angle 0°\ \Omega$ and is connected to a load of impedance $(400 + j300)\Omega$. Determine (a) the magnitude of the reflection coefficient and (b) the magnitude of the sending-end voltage if the reflected voltage is 14.60 V [(a) 0.345 (b) 42.32 V]

Standing-wave ratio

17 A transmission line has a characteristic impedance of $500\angle 0°\ \Omega$ and negligible loss. If the terminating impedance of the line is $(320 + j200)\Omega$ determine (a) the reflection coefficient and (b) the standing-wave ratio. [(a) $0.319\angle -61.72°$ (b) 1.937]

18 A low-loss transmission line has a mismatched load such that the reflection coefficient at the termination is $0.5\angle -135°$. The characteristic impedance of the line is 60 Ω. Calculate (a) the standing-wave

ratio, (b) the load impedance, and (c) the incident current flowing if the reflected current is 25 mA.
[(a) 3 (b) $113.93\angle 43.32^\circ\ \Omega$ (c) 50 mA]

19 The standing-wave ratio on a mismatched line is calculated as 2.20. If the incident power arriving at the termination is 100 mW, determine the value of the reflected power.
[14.06 mW]

20 The termination of a coaxial cable may be represented as a 150 Ω resistance in series with a 0.20 μH inductance. If the characteristic impedance of the line is $100\angle 0^\circ\ \Omega$ and the operating frequency is 80 MHz, determine (a) the reflection coefficient and (b) the standing-wave ratio. [(a) $0.417\angle -138.35^\circ$ (b) 2.43]

21 A cable has a characteristic impedance of $70\angle 0^\circ\ \Omega$. The cable is terminated by an impedance of $60\angle 30^\circ\ \Omega$. Determine the ratio of the maximum to minimum current along the line. [1.77]

from which, $$\ln v_C = -\frac{t}{CR} + k \qquad (45.7)$$

where k is a constant.

At time $t = 0$ (i.e., at the instant of opening the switch), $v_C = V$

Substituting $t = 0$ and $v_C = V$ in equation (45.7) gives:

$$\ln V = 0 + k$$

Substituting $k = \ln V$ into equation (45.7) gives:

$$\ln v_C = -\frac{t}{CR} + \ln V$$

and $$\ln v_C - \ln V = -\frac{t}{CR}$$

$$\ln \frac{v_C}{V} = -\frac{t}{CR}$$

and $$\frac{v_C}{V} = e^{-t/CR}$$

from which, $$\boxed{\boldsymbol{v_C = Ve^{-t/CR}}} \qquad (45.8)$$

i.e., the capacitor voltage, v_C, decays to zero after a period of time, the rate of decay depending on CR, which is the **time constant, τ** (see Section 17.3, page 260). Since $v_R + v_C = 0$ then the magnitude of the resistor voltage, v_R, is given by:

$$\boxed{\boldsymbol{v_R = Ve^{-t/CR}}} \qquad (45.9)$$

and since $i = C\frac{dv_C}{dt} = C\frac{d}{dt}(Ve^{-t/CR}) = (CV)\left(-\frac{1}{CR}\right)e^{-t/CR}$

i.e., the magnitude of the current, $$\boxed{\boldsymbol{i = \frac{V}{R}e^{-t/CR}}} \qquad (45.10)$$

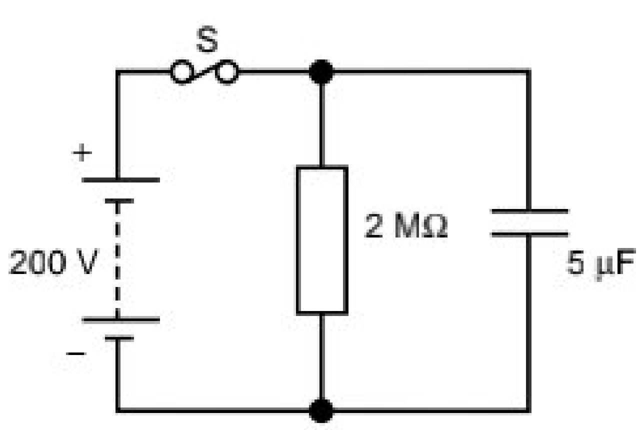


Figure 45.5

> Problem 2. A d.c. voltage supply of 200 V is connected across a 5 μF capacitor as shown in Figure 45.5. When the supply is suddenly cut by opening switch S, the capacitor is left isolated except for a parallel resistor of 2 MΩ. Calculate the p.d. across the capacitor after 20 s.

From equation (45.8), $v_C = Ve^{-t/CR}$

After 20 s, $v_C = 200e^{-20/(5\times10^{-6}\times2\times10^{6})} = 200\ e^{-2} = 200(0.13534)$

$= \mathbf{27.07\ V}$